Solutions to Exercises
Chapter 7: Extremal set theory

1 Verify the claim in Example 2 of Section 7.1.

$\mathcal{B}$ is an intersecting family, by inspection; so $\mathcal{B} \subseteq \mathcal{F}$. A 4-set contains at most one member of $\mathcal{B}$; so there $7 \cdot 4 = 28$ 4-sets containing members of $\mathcal{B}$ (as each 3-set lies in four 4-sets). Every 5-set contains a member of $\mathcal{B}$, since at most $3 \cdot 2 = 6$ members of $\mathcal{B}$ meet its complement. *A fortiori*, each 6-set or 7-set contains a member of $\mathcal{B}$. So

$$|\mathcal{F}| = 7 + 28 + 21 + 7 + 1 = 64.$$

2 If $n = 2k$, an intersecting family of k-subsets of an n-set has size at most $\frac{1}{2}\binom{n}{k} = \binom{n-1}{k-1}$, because it contains at most one of each complementary pair of k-sets. We proceed to generalise this result and argument. What follows could be regarded as a very simple version of the LYM technique. PROVE:

> *Suppose that k divides n. Then an intersecting family $\mathcal{F}$ of k-subsets of an n-set X has size at most $\binom{n-1}{k-1}$.*

Follow the hint. Let $\mathcal{C}$ be the set of all partitions of an n-set into n/k sets of size k, where k divides n. Count pairs (B,C) with B a k-set and $B \in C \in \mathcal{C}$. By symmetry, each of the $\binom{n}{k}$ choices for B lies in the same number, say x, of members of $\mathcal{C}$. On the other hand, each member of $\mathcal{C}$ contains n/k k-sets. So

$$\binom{n}{k} x = |\mathcal{C}|(n/k),$$

so $x = |\mathcal{C}|/\binom{n-1}{k-1}$, as claimed.

Now let $\mathcal{F}$ be an intersecting family of k-sets. Count pairs (B,C) with $B \in \mathcal{F}$, $C \in \mathcal{C}$, and $B \in C$. Each of the $|\mathcal{F}|$ choices for B lies in x choices for C, with x as above. Since $\mathcal{F}$ is intersecting, each of the $|\mathcal{C}|$ members of $\mathcal{C}$ contains at most one member of $\mathcal{F}$. Substituting for x gives the displayed inequality, and hence that $|\mathcal{F}| \le \binom{n-1}{k-1}$, as required.

Note that the bound is attained if and only if every member of $\mathcal{C}$ contains a member of $\mathcal{F}$.

3 Prove that, if k divides n and $n \ge 3k$, then any intersecting family of size $\binom{n-1}{k-1}$ of k-subsets of the n-set X consists of all k-sets containing some point of X.

First, we show that there are two sets $A,B \in \mathcal{F}$ which intersect in just one point. For suppose that the smallest intersection of two sets in $\mathcal{F}$ has size l, and suppose for a contradiction that $l > 1$. Let $|A \cap B| = l$. Choose two disjoint k-sets U,V such that

- U contains one point of $A \cap B$, all of $A \setminus B$, and none of $B \setminus A$;
- V contains the remaining points of $A \cap B$, all of $B \setminus A$, and none of $A \setminus B$.

Choose a partition C of X into k-sets including U and V. Then C must contain an element of $\mathcal{F}$, necessarily either U or V (since the other sets in C are disjoint from A and B). But $|U \cap B| = 1$ and $|V \cap A| = l-1$, contradicting the minimality of $|A \cap B|$.

Now assume that $n \geq 3k$. Let $A,B \in \mathcal{F}$ with $A \cap B = \{x\}$. We show that every set containing x is in $\mathcal{F}$; this will finish the proof, since $\binom{n-1}{k-1}$ sets contain x, which is just the right number. So let U be any set containing x. Choose two k-sets V,W disjoint from one another and from U, such that

- V contains all of $A \setminus U$ and none of $B \setminus U$;
- W contains all of $B \setminus U$ and none of $A \setminus U$;

Choose a partition C of X into k-sets including U,V,W. Then C contains a member of $\mathcal{F}$, necessarily U (since V is disjoint from B, W from A, and the other sets in C are disjoint from both). That is, $U \in \mathcal{F}$, as required.

4 Show that $\binom{2k-1}{k-1}$ is even if and only if k is not a power of 2.

Use Lucas' Theorem (3.4.1). Let $k-1$ have digits $a_l a_{l-1} \ldots a_1$ in base 2, with $a_l = 1$. Then $2k-1$ has digits $a_l a_{l-1} \ldots a_0 1$ (it is obtained from $k-1$ by multiplying by 2 and adding 1). By Lucas's Theorem,

$$\binom{2k-1}{k-1} \equiv \prod_{i=0}^{l} \binom{a_{i-1}}{a_i} \pmod 2,$$

with the convention that $a_{-1} = 1$, $a_{l+1} = 0$. If there is a value of i such that $a_{i-1} = 0$ and $a_i = 1$, then the right-hand side has a factor $\binom{0}{1} = 0$, and so is even. So, if $\binom{2k-1}{k-1}$ is odd, then $(a_i = 1) \Rightarrow (a_{i-1} = 1$. Since $a_l = 1$, we conclude that $a_{l-1} = \ldots = a_0 = 1$. Thus

$$k-1 = 2^l + 2^{l-1} + \cdots + 1 = 2^{l+1} - 1,$$

and k is a power of 2, as claimed.

5 (a) If n is not a power of 2, construct a regular intersecting family of subsets of an n-set, having size 2^{n-1}.

(b) If $n = 2, 4$ or 8, show that there is no such family.

(a) Suppose that $n = 2k$ is not a power of 2. By (7.4.2), there is a regular intersecting family of k-subsets of an n-set, containing one of each complementary pair of k-sets. Adjoin to it all sets of cardinality greater than k. The resulting family is still intersecting, is regular (since the m-sets form a regular family for any m) and has cardinality 2^{n-1} (since it contains one of each complementary pair of sets of whatever size).

(b) Case analysis. The case $n = 2$ is trivial.

Let $n = 4$ and suppose that $\mathcal{F}$ contains one of each complementary pair of subsets. If $\mathcal{F}$ contains a singleton x, then it consists of all sets containing x, which is not a regular family. Otherwise, $\mathcal{F}$ contains all 3-sets and 4-sets, and a regular intersecting family of three 2-sets, which is impossible since the degree would have to be $3 \cdot 2/4$, not an integer.

The case $n = 8$ is a bit more complicated, so we begin with some observations. Let $\mathcal{F}$ be an intersecting family of size 2^{n-1}. Then, if $A \in \mathcal{F}$ and $A \subseteq B$, then also $B \in \mathcal{F}$. For $\mathcal{F}$ contains either B or its complement; and the latter is disjoint from A.

In the case $n = 8$, a family consisting of the largest $2^7 = 128$ sets (viz. all of size $8, 7, 6, 5$, and 35 of size 4) has average degree

$$(8 + 8 \cdot 7 + 28 \cdot 6 + 56 \cdot 5 + 35 \cdot 4)/8 = 81.5,$$

so a regular intersecting family has degree at most 81. Hence it cannot contain a singleton: if $\{x\} \in \mathcal{F}$, then x lies in 128 sets). Also, it has at most one set of size 2: for, if $\{x,y\}, \{x,z\} \in \mathcal{F}$, then x is contained in 96 sets which also contain y or z.

The remainder of the argument involves case analysis.

6 Prove that, in any intersecting family of size $\binom{2k-1}{k-1}$ of k-subsets of a $2k$-set, the replication numbers all have the same parity.

6. Replacing one set in such a family by its complement changes the parity of all the replication numbers. By a sequence of such switches, we can move from any such family to any other. So it is enough to show that there is a family in which all the replication numbers have the same parity. See the proof of (7.4.2) for this.

7 Let $\mathcal{F}$ be any intersecting family of subsets of the n-set X. Show that there is an intersecting family $\mathcal{F}' \supseteq \mathcal{F}$ with $|\mathcal{F}'| = 2^{n-1}$.

Let (Y,Z) be any partition of X. Then at most one of Y and Z can contain an element of $\mathcal{F}$. If neither of them does, then both Y and Z are blocking sets. So, if we let $\mathcal{F}'$ consist of all sets containing a member of $\mathcal{F}$ together with one of each complementary pair of blocking sets, then $\mathcal{F}'$ contains one of each complementary pair of sets, and so $|\mathcal{F}'| = 2^{n-1}$.

Furthermore, since $\mathcal{F}$ is intersecting, then any two sets which contain members of $\mathcal{F}$ must intersect; a set containing a member of $\mathcal{F}$ intersects each blocking set; and, if we choose the larger of each pair of blocking sets (as in the question), then any two of the chosen blocking sets intersect. So we have an intersecting family containing $\mathcal{F}$.

Let $\mathcal{F}$ be the Steiner triple system. We saw in Question 1 that any set of 5 or more points contains a member of $\mathcal{F}$; so blocking sets have size 3 or 4, and it is enough to show that there are none of size 3. Let Y be a set of size 3, and let A_i be the set of members of $\mathcal{F}$ containing $i \in Y$. Then, with the notation of PIE, $|A_I| = 7,3,1$ for $|I| = 0,1,2$. So, if m sets contain all three points of Y, the number of sets containing none of them is

$$7 - 3\cdot 3 + 3\cdot 1 - m = 1 - m.$$

Thus, either $Y \in \mathcal{F}$, or there is a set of $\mathcal{F}$ disjoint from Y. In either case, Y is not a blocking set.

So the construction of the first part of the question produces just the sets which contain a member of $\mathcal{F}$; there are $2^6 = 64$ of them.

8 Let $\mathcal{F}$ be a Sperner family of subsets of the n-set X. Define $b(\mathcal{F})$ to be the family of all subsets Y of X such that

(i) $Y \cap F \neq \emptyset$ for all $F \in \mathcal{F}$;

(ii) Y is minimal subject to (i) (i.e., no proper subset of Y satisfies (i)).

(a) Prove that $b(\mathcal{F})$ is a Sperner family.

(b) Show that, for any $F \in \mathcal{F}$ and any $y \in F$, there exists $Y \in b(\mathcal{F})$ with $Y \cap F = \{y\}$.

(c) Deduce that $b(b(\mathcal{F})) = \mathcal{F}$.

(d) Let $\mathcal{F}_k$ denote the Sperner family of all k-subsets of X. Prove that $b(\mathcal{F}_k) = \mathcal{F}_{n+1-k}$ for $k > 0$. What is $b(\mathcal{F}_0)$?

(a) If one member of $b(\mathcal{F})$ properly contained another, the larger would not be minimal with respect to intersecting all the sets in $\mathcal{F}$.

(b) Take $F \in \mathcal{F}$ and $y \in F$. As in the Hint, let Z be minimal with respect to meeting every set $F' \setminus F$ for $F' \in \mathcal{F}$, $F' \neq F$. (Since $\mathcal{F}$ is a Sperner family, all these differences are non-empty.) Note that $Z \cap F = \emptyset$. Now $\{y\} \cup Z$ meets every member of $\mathcal{F}$, so it contains a set $Y \in b(\mathcal{F})$; and certainly $y \in Y$, since otherwise $Y \cap F = \emptyset$.

(c) Take $F \in \mathcal{F}$. Then F meets every set of $b(\mathcal{F})$. But, by (b), for every $y \in F$, $F \setminus \{y\}$ is disjoint from some member of $b(\mathcal{F})$. So no proper subset of F meets every member of $b(\mathcal{F})$. Thus F is minimal with respect to this property, that is, $F \in b(b(\mathcal{F}))$.

Conversely, take $G \in b(b(\mathcal{F}))$, and suppose that $G \notin \mathcal{F}$. Then G contains no member of $\mathcal{F}$, since $b(b(\mathcal{F}))$ is a Sperner family and contains $\mathcal{F}$. So the complement $X \setminus G$ meets every member of $\mathcal{F}$. Then $X \setminus G$ contains a member Y of $b(\mathcal{F})$, so that $Y \cap G = \emptyset$, a contradiction.

(d) If $k > 0$, then an $(n+1-k)$-set meets every k-set, whereas an $(n-k)$-set is disjoint from one k-set (viz., its complement). So $\mathcal{F}_{n+1-k} \subseteq b(\mathcal{F}_k)$. The inclusion cannot be strict, else $b(\mathcal{F}_k)$ would not be a Sperner family.

We have $\mathcal{F}_0 = \{\emptyset\}$. No set meets the empty set, so $b(\mathcal{F}_0) = \emptyset$. (Note the difference. Note also that, to calculate $b(\emptyset)$, the condition of meeting every member of $\emptyset$ is vacuous, so every set satisfies it; the unique minimal such set is $\emptyset$, so

$b(\emptyset) = \{\emptyset\}$, in agreement with (c).