The Rado graph and the Urysohn space

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Rado's graph

In 1964, Rado constructed a universal graph as follows: The vertex set is the set of natural numbers (including zero). For $i, j \in \mathbb{N}$, $i < j$, then i and j are joined if and only if the ith digit in j (in base 2) is 1.

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Another construction:
Let $\mathbb{P}_1$ denote the set of primes congruent to 1 mod 4.
According to the Quadratic Reciprocity Law, for $p, q \in \mathbb{P}_1$, p is a square mod q if and only if q is a square mod p. Join p to q if this holds.
This graph is isomorphic to Rado's.

Universality and homogeneity

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Exercise: Find an automorphism interchanging 0 and 1.

Uniqueness

Rado's graph is the unique (up to isomorphism) graph which is countable, universal and homogeneous.
In fact, it suffices for this statement to assume universality for finite graphs (that is, every finite graph can be embedded as an induced subgraph) and homogeneity.

Recognition

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Thus, Rado's graph is the unique countable graph (up to isomorphism) satisfying condition $(*)$.

Measure and category

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There is a complete metric on the set of graphs: the distance between two graphs is $1/2^n$ if n is minimal such that x_n and y_n are joined in one graph but not the other. Now a set of graphs is “large” if it is residual in the sense of Baire category, that is, contains a countable intersection of open dense sets.

Ubiquity

It is now quite easy to show that the set of countable graphs satisfying $(*)$ (that is, the set of graphs isomorphic to R) is "large" in both the senses just described.

In fact, condition $(*)$ with fixed sets U and V is satisfied in an open dense set of graphs with full measure, and there are only countably many choices of the pair (U, V).

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Thus, Rado's graph is *the* countable random graph, as well as *the* generic countable graph.

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- more generally, adding or deleting any set of edges such that only finitely many are incident with each vertex;
- taking the complement.

Pigeonhole property

A countable graph G is said to have the pigeonhole property if, whenever the vertex set of G is partitioned into two parts in any manner, the induced subgraph on one of these parts is isomorphic to G.

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Rado's graph has the pigeonhole property.

Indeed, there are just three countable graphs with the pigeonhole property: the complete graph, the null graph, and Rado's graph.

Spanning subgraphs

A countable graph G is a spanning subgraph of R if and only if, for any finite set W of vertices of G, there is a vertex Z joined to no vertex in W.
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Dually, R is a spanning subgraph of G if and only if any finite set of vertices of G have a common neighbour.

Factorisations

Theorem

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Proof.

Enumerate the edges of R: $e_1, e_2, \dots$. Suppose we have found disjoint subgraphs $G'_1, \dots, G'_n$ isomorphic to $G_1, \dots, G_n$ and containing $e_1, \dots, e_n$. Then $R \setminus (G'_1 \cup \dots \cup G'_n)$ is isomorphic to R, so contains a spanning subgraph G'_{n+1} isomorphic to G_{n+1}; moreover, since the automorphism group of R is edge-transitive, we may assume that this subgraph contains e_{n+1}, if this edge is not already covered by $G'_1, \dots, G'_n$. □

Automorphisms

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- $\mathrm{Aut}(R)$ contains a generic conjugacy class, one that is residual in the whole group;
- $\mathrm{Aut}(R)$ contains a copy of every finite or countable group.

Homomorphisms

A homomorphism of a graph G is a map from G to G which maps edges to edges. The endomorphisms of any graph G (the homomorphisms from G to G) form a monoid (a semigroup with identity).
The endomorphism monoid of R contains a copy of every finite or countable monoid.

Homomorphism-homogeneity

Recall that a graph G is homogeneous if every isomorphism between finite subgraphs of G can be extended to an isomorphism from G to G.
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What is known?

- Every graph containing R as a spanning subgraph is homomorphism- and monomorphism-homogeneous.
- If a countable graph G has the property that every monomorphism between finite subgraphs extends to a homomorphism of G, then either G contains R as a spanning subgraph, or there is a bound on the size of claws $K_{1,n}$ in G.

Apart from disjoint unions of complete graphs (which contain no $K_{1,2}$), no homomorphism-homogeneous graphs of bounded claw size are known.

Polish spaces

There is a complete metric space with properties remarkably similar to those of Rado's graph.

A complete metric space will not usually be countable. Instead we require it to be separable, that is, to have a countable dense subset.

A Polish space is a complete separable metric space.

Thus, the completion of any countable metric space is a Polish space. (This is analogous to the construction of $\mathbb{R}$ from $\mathbb{Q}$.)

Urysohn space

In a posthumous paper published in 1927, P. S. Urysohn showed:

Theorem

There is a unique Polish space which is

- *universal, that is, every Polish space can be isometrically embedded into it;*
- *homogeneous, that is, every isometry between finite subsets can be extended to an isometry of the whole space.*

We denote Urysohn space by $\mathbb{U}$.

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Suppose that points $a_1, \ldots, a_n$ have been constructed and their distances $d(a_i, a_j)$ specified. We want to add a new point a_{n+1} with distances $d(a_{n+1}, a_i) = x_i$ for $i = 1, \ldots, n$. These distances must satisfy $x_i \geq 0$ for $i = 1, \ldots, n$ and

$$|x_i - x_j| \leq d(a_i, a_j) \leq x_i + x_j$$

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$$|x_i - x_j| \leq d(a_i, a_j) \leq x_i + x_j$$

for $i, j = 1, \ldots, n$.
Thus the possible distances are chosen from a cone in $\mathbb{R}^n$.

Ubiquity

Thus we have both a measure and a metric on the set of countable metric spaces. For the measure, use any natural probability measure on the cone in $\mathbb{R}^n$ at each step, for example, the restriction of a Gaussian measure on the whole space.

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- the completion of a random countable metric space is isometric to $\mathbb{U}$ with probability 1;
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In other words, Urysohn space is *the* random Polish space, and *the* generic Polish space.
Unfortunately we don't have a simple direct construction of $\mathbb{U}$.

Rado and Urysohn

Any countable dense subset of $\mathbb{U}$ carries the structure of Rado's graph R (in many different ways). Simply partition the set of distances which occur into two subsets E and N (satisfying some weak restrictions), and join x to y if $d(x, y) \in E$.

Rado and Urysohn

Any countable dense subset of $\mathbb{U}$ carries the structure of Rado's graph R (in many different ways). Simply partition the set of distances which occur into two subsets E and N (satisfying some weak restrictions), and join x to y if $d(x, y) \in E$. Hence, if a group G acts as an isometry group of $\mathbb{U}$ with a countable dense orbit, then G acts as an automorphism group of R.

Examples

The Urysohn space admits an isometry all of whose orbits are dense. So the infinite cyclic group is an example of a group acting on R. (In fact, if we choose a "random countable circulant graph", it is isomorphic to R with probability 1.

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The reverse implication is false. The countable elementary abelian 3-group acts on R but not on $\mathbb{U}$.

Ramsey theory

There is a close connection between homogeneity and Ramsey theory.
Hubicka and Nešetřil have shown that, if a countably infinite structure carries a total order and the class of its finite substructures is a Ramsey class, then the infinite structure is homogeneous.
The finite substructures of R are the finite graphs, which do form a Ramsey class.
The converse is false in general, but Nešetřil recently showed that the class of finite metric spaces is a Ramsey class.