# Sudoku – an alternative history

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# Sudoku

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Instructions in *The Independent*

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> *There's no mathematics involved. Use logic and reasoning to solve the puzzle.*
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But who invented Sudoku?

- Leonhard Euler
- W. U. Behrens
- John Nelder
- Howard Garns
- Robert Connelly

# Euler

Euler posed the following question in 1782.

> *Of 36 officers, one holds each combination of six ranks and six regiments. Can they be arranged in a* $6 \times 6$ *square on a parade ground, so that each rank and each regiment is represented once in each row and once in each column?*

# NO!!

# But he could have done it with 16 officers . . .

(thanks to Liz McMahon and Gary Gordon)

# Why was Euler interested?

A magic square is an $n \times n$ square containing the numbers $1, \ldots, n^2$ such that all rows, columns, and diagonals have the same sum.

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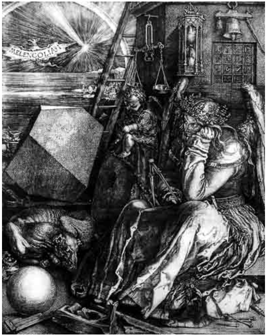

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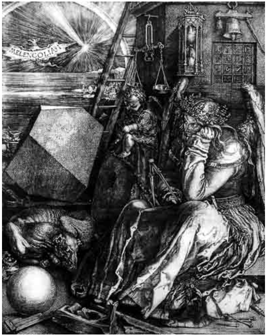

| | | | |
|---|---|---|---|
| 16 | 3 | 2 | 13 |
| 5 | 10 | 11 | 8 |
| 9 | 6 | 7 | 12 |
| 4 | 15 | 14 | 1 |

# Euler's construction

Suppose we have a solution to Euler's problem with $n^2$ officers in an $n \times n$ square. Number the regiments and the ranks from 0 to $n-1$; then each officer is represented by a 2-digit number in base $n$, in the range $0 \ldots n^2 - 1$. Add one to get the range $1 \ldots n^2$. It is easy to see that the row and column sums are constant. A bit of rearrangement usually makes the diagonal sums constant as well.

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| | | |
|---|---|---|
| $C\beta$ | $A\gamma$ | $B\alpha$ |
| $A\alpha$ | $B\beta$ | $C\gamma$ |
| $B\gamma$ | $C\alpha$ | $A\beta$ |

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|---|---|---|
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| $A\alpha$ | $B\beta$ | $C\gamma$ |
| $B\gamma$ | $C\alpha$ | $A\beta$ |

| | | |
|---|---|---|
| 21 | 01 | 10 |
| 00 | 11 | 22 |
| 12 | 20 | 01 |

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| | | |
|---|---|---|
| 21 | 01 | 10 |
| 00 | 11 | 22 |
| 12 | 20 | 01 |

| | | |
|---|---|---|
| 8 | 3 | 4 |
| 1 | 5 | 9 |
| 6 | 7 | 2 |

# Latin squares

A Latin square of order $n$ is an $n \times n$ array containing the symbols $1, \ldots, n$ such that each symbol occurs once in each row and once in each column. The name was invented by the statistician R. A. Fisher in the twentieth century, as a back-formation from "Graeco-Latin square" in the case where we have only one set of symbols.

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The Cayley table of a group is a Latin square. In fact, the Cayley table of a binary system $(A, \circ)$ is a Latin square if and only if $(A, \circ)$ is a quasigroup. (This means that left and right division are uniquely defined, i.e. the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions $x$ and $y$ for any $a$ and $b$.)

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Example

| $\circ$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $b$ | $a$ | $c$ |
| $b$ | $a$ | $c$ | $b$ |
| $c$ | $c$ | $b$ | $a$ |

# About Latin squares

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- There is a Markov chain method to choose a random Latin square. But we don't know much about what a random Latin square looks like.
- For example, the second row is a permutation of the first; this permutation is a derangement (i.e. has no fixed points). Are all derangements roughly equally likely?

# Orthogonal Latin squares

Two Latin squares $A$ and $B$ are orthogonal if, given any $k, l$, there are unique $i, j$ such that $A_{ij} = k$ and $B_{ij} = l$.

Thus, a Graeco-Latin square is a pair of orthogonal Latin squares.

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But we don't know

- how many orthogonal pairs of Latin squares of order $n$ there are;
- the maximum number of mutually orthogonal Latin squares of order $n$;
- how to choose at random an orthogonal pair.

# Projective planes

A projective plane is a geometry of points and lines such that any two points lie on a unique line and any two lines intersect in a unique point (together with a non-degeneracy condition to rule out trivial cases: there should exist four points with no three collinear).

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A finite projective plane has $n^2 + n + 1$ points and the same number of lines, for some integer $n > 1$ called the order of the plane.
A projective plane of order $n$ exists if and only if there are $n - 1$ pairwise orthogonal Latin squares of order $n$.
It is known that there is a projective plane of any prime power order, and that there is none of order 6 or 10. (The latter non-existence result comes from a huge computation by Clement Lam and others.)

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This encrypts a string of symbols in a fixed alphabet. It requires a key, a random string of the same length in the same alphabet, and an encryption table, a Latin square with rows and columns labelled by the alphabet.

To encrypt data symbol $x$ with key symbol $y$, we look in row $x$ and column $y$ of the encryption table, and put the symbol $z$ in this cell in the ciphertext.

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In the Second World War, the Japanese navy used this system with alphabet $\{0, \ldots, 9\}$. Sometimes their encryption tables failed to be Latin squares.

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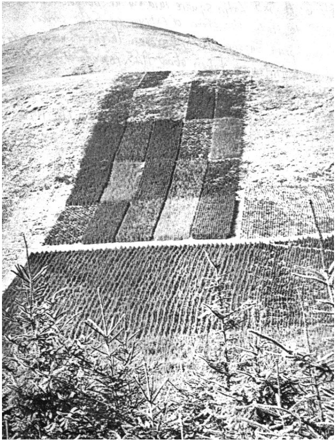

A Latin square in Beddgelert Forest, designed by R. A. Fisher.

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Take an $n \times n$ grid divided into $n$ regions, with $n$ cells in each. A gerechte design for this partition involves filling the cells with the numbers $1, \ldots, n$ in such a way that each row, column, or region contains each of the numbers just once. So it is a special kind of Latin square.

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## Example

Suppose that there is a boggy patch in the middle of the field.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 4 | 5 | 1 | 2 | 3 |
| 2 | 3 | 4 | 5 | 1 |
| 5 | 1 | 2 | 3 | 4 |
| 3 | 4 | 5 | 1 | 2 |

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How difficult is it to recognise a critical set, or to complete one?

# Garns

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A Sudoku puzzle is a critical set for a gerechte design for the $9 \times 9$ grid partitioned into $3 \times 3$ subsquares. The puzzler's job is to complete the square.

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Garns called his puzzle "number place". It became popular in Japan under the name "Sudoku" in 1986 and returned to the West a couple of years ago.

# Connelly

Robert Connelly proposed a variant which he called symmetric Sudoku. The solution must be a gerechte design for all these regions:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 9 | 2 | 4 | 8 | 1 | 6 | 7 |
| 4 | 8 | 1 | 6 | 7 | 3 | 5 | 9 | 2 |
| 7 | 2 | 6 | 9 | 1 | 5 | 8 | 3 | 4 |
| 8 | 1 | 4 | 7 | 3 | 6 | 9 | 2 | 5 |
| 2 | 6 | 7 | 1 | 5 | 9 | 3 | 4 | 8 |
| 5 | 9 | 3 | 4 | 8 | 2 | 6 | 7 | 1 |
| 6 | 7 | 2 | 5 | 9 | 1 | 4 | 8 | 3 |
| 9 | 3 | 5 | 8 | 2 | 4 | 7 | 1 | 6 |
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| 4 | 8 | 1 | 6 | 7 | 3 | 5 | 9 | 2 |
| 7 | 2 | 6 | 9 | 1 | 5 | 8 | 3 | 4 |
| 8 | 1 | 4 | 7 | 3 | 6 | 9 | 2 | 5 |
| 2 | 6 | 7 | 1 | 5 | 9 | 3 | 4 | 8 |
| 5 | 9 | 3 | 4 | 8 | 2 | 6 | 7 | 1 |
| 6 | 7 | 2 | 5 | 9 | 1 | 4 | 8 | 3 |
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Rows

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 9 | 2 | 4 | 8 | 1 | 6 | 7 |
| 4 | 8 | 1 | 6 | 7 | 3 | 5 | 9 | 2 |
| 7 | 2 | 6 | 9 | 1 | 5 | 8 | 3 | 4 |
| 8 | 1 | 4 | 7 | 3 | 6 | 9 | 2 | 5 |
| 2 | 6 | 7 | 1 | 5 | 9 | 3 | 4 | 8 |
| 5 | 9 | 3 | 4 | 8 | 2 | 6 | 7 | 1 |
| 6 | 7 | 2 | 5 | 9 | 1 | 4 | 8 | 3 |
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Columns

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 9 | 2 | 4 | 8 | 1 | 6 | 7 |
| 4 | 8 | 1 | 6 | 7 | 3 | 5 | 9 | 2 |
| 7 | 2 | 6 | 9 | 1 | 5 | 8 | 3 | 4 |
| 8 | 1 | 4 | 7 | 3 | 6 | 9 | 2 | 5 |
| 2 | 6 | 7 | 1 | 5 | 9 | 3 | 4 | 8 |
| 5 | 9 | 3 | 4 | 8 | 2 | 6 | 7 | 1 |
| 6 | 7 | 2 | 5 | 9 | 1 | 4 | 8 | 3 |
| 9 | 3 | 5 | 8 | 2 | 4 | 7 | 1 | 6 |
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Subsquares

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|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 9 | 2 | 4 | 8 | 1 | 6 | 7 |
| 4 | 8 | 1 | 6 | 7 | 3 | 5 | 9 | 2 |
| 7 | 2 | 6 | 9 | 1 | 5 | 8 | 3 | 4 |
| 8 | 1 | 4 | 7 | 3 | 6 | 9 | 2 | 5 |
| 2 | 6 | 7 | 1 | 5 | 9 | 3 | 4 | 8 |
| 5 | 9 | 3 | 4 | 8 | 2 | 6 | 7 | 1 |
| 6 | 7 | 2 | 5 | 9 | 1 | 4 | 8 | 3 |
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Broken rows

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| 7 | 2 | 6 | 9 | 1 | 5 | 8 | 3 | 4 |
| 8 | 1 | 4 | 7 | 3 | 6 | 9 | 2 | 5 |
| 2 | 6 | 7 | 1 | 5 | 9 | 3 | 4 | 8 |
| 5 | 9 | 3 | 4 | 8 | 2 | 6 | 7 | 1 |
| 6 | 7 | 2 | 5 | 9 | 1 | 4 | 8 | 3 |
| 9 | 3 | 5 | 8 | 2 | 4 | 7 | 1 | 6 |
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Broken columns

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|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 9 | 2 | 4 | 8 | 1 | 6 | 7 |
| 4 | 8 | 1 | 6 | 7 | 3 | 5 | 9 | 2 |
| 7 | 2 | 6 | 9 | 1 | 5 | 8 | 3 | 4 |
| 8 | 1 | 4 | 7 | 3 | 6 | 9 | 2 | 5 |
| 2 | 6 | 7 | 1 | 5 | 9 | 3 | 4 | 8 |
| 5 | 9 | 3 | 4 | 8 | 2 | 6 | 7 | 1 |
| 6 | 7 | 2 | 5 | 9 | 1 | 4 | 8 | 3 |
| 9 | 3 | 5 | 8 | 2 | 4 | 7 | 1 | 6 |
| 1 | 4 | 8 | 3 | 6 | 7 | 2 | 5 | 9 |

Locations

# Coordinates

We coordinatise the cells of the grid with $F^4$, where $F$ is the integers mod 3, as follows:

- the first coordinate labels large rows;
- the second coordinate labels small rows within large rows;
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Now Connelly's regions are cosets of the following subspaces:

| | | | |
|---|---|---|---|
| Rows | $x_1 = x_2 = 0$ | Columns | $x_3 = x_4 = 0$ |
| Subsquares | $x_1 = x_3 = 0$ | Broken rows | $x_2 = x_3 = 0$ |
| Broken columns | $x_1 = x_4 = 0$ | Locations | $x_2 = x_4 = 0$ |

# Affine spaces

Let $F$ be the field of integers mod 3. As we saw, the four-dimensional affine space over $F$ has point set $F^4$.

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A line is the set of points satisfying three independent linear equations, or equivalently the set of points of the form $x = a + \lambda b$ for fixed $a, b \in F^4$, where $\lambda$ runs through $F$. Note that, if $b_i = 0$, then $x_i = a_i$ for any point $x$, while if $b_i \neq 0$, then $x_i$ runs through the three values in $F$.

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Conversely, a set of three points which are either constant or take all values in each coordinate is a line.

# Affine spaces and SET®

The card game SET has 81 cards, each of which has four attributes taking three possible values (number of symbols, shape, colour, and shading). A winning combination is a set of three cards on which either the attributes are all the same, or they are all different.

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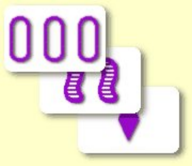

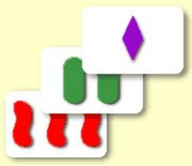

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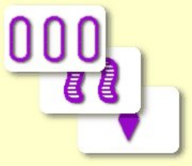 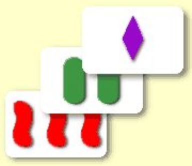

Each card has four coordinates taken from $F$ (the integers mod 3), so the set of cards is identified with the 4-dimensional affine space. Then the winning combinations are precisely the affine lines!

# Perfect codes

A code is a set $C$ of "words" or $n$-tuples over a fixed alphabet $F$. The Hamming distance between two words $v, w$ is the number of coordinates where they differ; that is, the number of errors needed to change the transmitted word $v$ into the received word $w$.

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A code $C$ is $e$-error-correcting if there is *at most* one word at distance $e$ or less from any codeword. [Equivalently, any two codewords have distance at least $2e + 1$.] We say that $C$ is perfect $e$-error-correcting if "at most" is replaced here by "exactly".

# Perfect codes and symmetric Sudoku

Take a solution to a symmetric Sudoku puzzle, and look at the set $S$ of positions of a particular symbol $s$. The coordinates of the points of $S$ have the property that any two differ in at least three places; that is, they have Hamming distance at least 3. [For, if two of these words agreed in the positions 1 and 2, then $s$ would occur twice in a row; and similarly for the other pairs.]

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So a symmetric Sudoku solution is a partition of $F^4$ into nine perfect codes.

# All symmetric Sudoku solutions

Now it can be shown that a perfect code $C$ in $F^4$ is an affine plane, that is, a coset of a 2-dimensional subspace of $F^4$. To show this, we use the SET® principle: We show that if $v, w \in C$, then the word which agrees with $v$ and $w$ in the positions where they agree and differs from them in the positions where they differ is again in $C$.

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It is not hard to show that there are just two ways to do this.

One solution consists of nine cosets of a fixed subspace.

There is just one further type, consisting of six cosets of one subspace and three of another. [Take a solution of the first type, and replace three affine planes in a 3-space with a different set of three affine planes.]

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An earlier computation by Felgenhauer and Jarvis gives the total number of solutions to be 6 670 903 752 021 072 936 960. Now for each conjugacy class of non-trivial symmetries of the grid, it is somewhat easier to calculate the number of fixed solutions.

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The last two questions are particularly interesting in the case where $n = kl$ and the regions are $k \times l$ rectangles.

# References


- R. A. Bailey, P. J. Cameron and R. Connelly, Sudoku, Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes, *American Math. Monthly*, to appear. Preprint available from `http://www.maths.qmul.ac.uk/~pjc/preprints/sudoku.pdf`