The profile of a relational structure

Peter J. Cameron
Queen Mary, University of London
Isaac Newton Institute, Cambridge

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The profile is the sequence $(f_0, f_1, f_2, \ldots)$, were f_n is the number of n-element structures in the age, up to isomorphism.

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  - Profile: $f_n \sim 2^{n(n-1)/2}/n!$

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So the profile of R also counts orbits of G on n-element subsets of Ω for $n = 0, 1, 2, \ldots$.

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- In the latter case, $f_n \geq \exp(n^{1/2-\epsilon})$ for sufficiently large n. (These two results assume that the number of relations is finite).
- In the case of a primitive permutation group (one preserving no non-trivial equivalence relation), there is a constant $c > 1$ such that either $f_n = 1$ for all n, or $f_n \geq c^n/p(n)$ for some polynomial p.

Local conditions

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There are two known proofs of this theorem; one using a Ramsey-type theorem (outlined on the next slide), the other using finite combinatorics and linear algebra (see later).

A Ramsey-type theorem

Given a colouring of the n-sets with colours $c_1, \ldots, c_r$, we say that the *colour scheme* of an $(n+1)$-set S is the r-tuple $(a_1, \ldots, a_r)$, where a_i is the number of sets of colour c_i in S.

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Let the n-subsets of an infinite (or sufficiently large finite) set Ω be coloured with r colours (all of which are used). Then there are at least r colour schemes of $(n+1)$-sets. In fact, there exist $(n+1)$-sets $T_1, \ldots, T_r$ so that T_i contains a set of colour c_i but none of colour c_j for $j > i$.

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The "Ramsey numbers" associated with this theorem are not known.

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There is a multiplication defined on $A = \bigoplus_{n \geq 0} V_n$ as follows: for $f \in V_n$, $g \in V_m$, and $X \in \binom{\Omega}{m+n}$, put

$$(fg)(X) = \sum_{Y \in \binom{X}{n}} f(Y) g(X \setminus Y).$$

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In the fourth of our examples, A is the shuffle algebra on k symbols.

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This theorem is proved by finite combinatorial arguments. It implies that multiplication by e is a monomorphism from V_n to V_{n+1}, and hence

$$f_n = \dim(V_n) \leq \dim(V_{n+1}) = f_{n+1}$$

for any n.

Two conjectures

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The first of these conjectures has very recently been proved by Maurice Pouzet.

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Theorem

Assume that R is inexhaustible. Then $f_{m+n} \geq f_m + f_n - 1$.

In outline: multiplication induces a map from the *Segre variety* (the rank 1 tensors modulo scalars) in $V_m \otimes V_n$ into V_{m+n} modulo scalars; so the dimension of V_{m+n} is at least as great as that of the Segre variety.

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In a similar way, if the second part of the conjecture is true, then the profile of an inexhaustible structure would satisfy $g_{m+n} \geq g_m + g_n - 1$, where $g_n = f_{n+1} - f_n$. (Apply a similar argument to $A/\langle e \rangle$, whose nth homogeneous component is V_{n+1}/eV_n, with dimension $f_{n+1} - f_n$.)

Sketch proof

Let Ω be an set, $\mathbb{K}$ a field with characteristic zero. Let $f : \binom{\Omega}{n} \to \mathbb{K}$. The support of f is $\{X \in \binom{\Omega}{n} : f(X) \neq 0\}$. A set T is a transversal to a family $\mathcal{H}$ of sets if $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$. The transversality of $\mathcal{H}$ is the cardinality of the smallest transversal.

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Pouzet proved:

Theorem

Given $m, n \geq 0$, there exists t such that, for any Ω with $|\Omega| \geq m + n$, any field $\mathbb{K}$ of characteristic zero, and any two non-zero maps $f : \binom{\Omega}{n} \to \mathbb{K}$, $g : \binom{\Omega}{m} \to \mathbb{K}$ such that $fg = 0$, the transversality of $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ is at most t.

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The result follows since removal of a transversal would decrease the age, which is impossible in an inexhaustible structure.

Ramsey numbers

The theorem is a Ramsey-type theorem, and one can ask for an evaluation of $\tau(m, n)$, the smallest number t for which the conclusion of the theorem is true. It is not hard to show that $\tau(1, n) = 2n$: this is the combinatorics underlying the proof that $f_n \leq f_{n+1}$.

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Pouzet's proof shows that

$$7 \leq \tau(2,2) \leq 2(R_k^2(4) + 2),$$

where $k = 5^{30}$ and $R_k^2(4)$ is the classical Ramsey number, the least p such that in any k-colouring of the edges of the complete graph on p vertices, there is a monochromatic subgraph of order 4.

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This is rather a large gap – can it be reduced?

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A more interesting possibility involves showing that, under suitable hypotheses to be determined, if $f_1, \ldots, f_r \in V_n$ and $g_1, \ldots, g_r \in V_m$ are linearly independent, then

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If this were true, the dimension argument would give a much stronger lower bound for f_{m+n} in terms of f_m and f_n.

But it cannot be true in general since the earlier bound is tight in some cases!