Synchronization and homomorphisms

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July 2008

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We can represent an automaton as an edge-coloured directed graph, where the vertices are the states, and the colours are the transitions. We require that the graph should have exactly one edge of each colour *leaving* each vertex.

Synchronization

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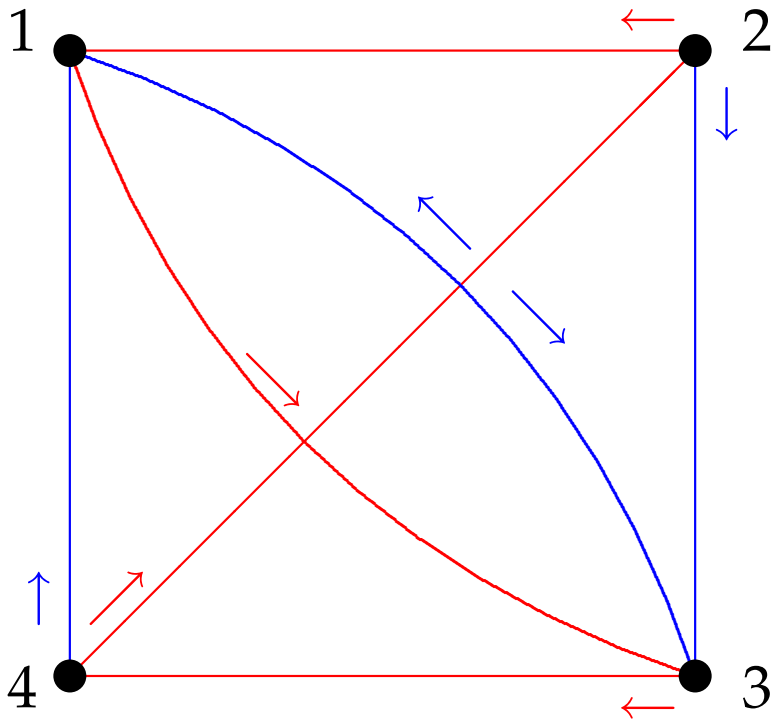
A reset word is a sequence of transitions which take the automaton from any state into a known state; in other words, the composition of the corresponding transitions is a constant mapping.

An example



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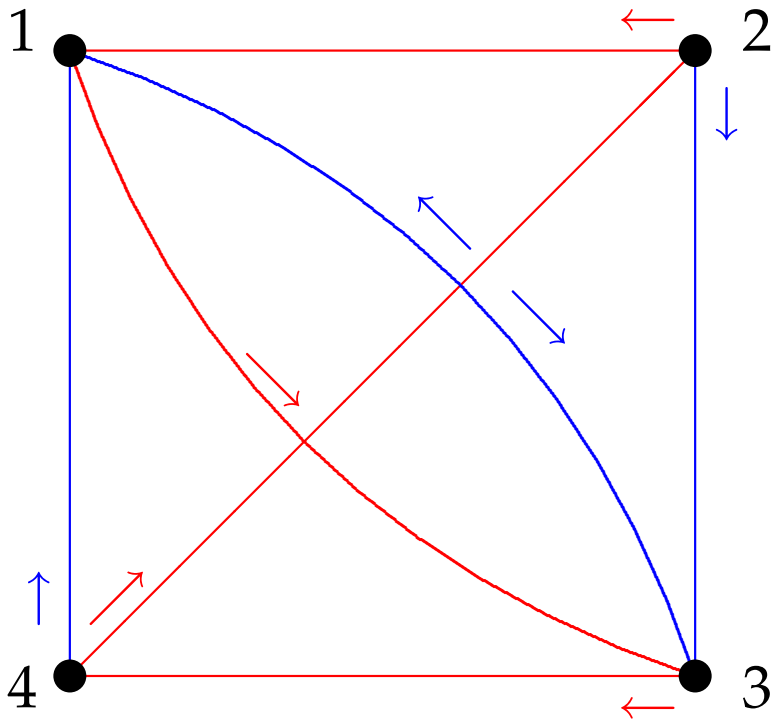

You can check that (Blue, Red, Blue, Blue) is a reset word which takes you to state 3 no matter where you start.

Applications

- Industrial robotics: pieces arrive to be assembled by a robot. The orientation is critical. You could equip the robot with vision sensors and manipulators so that it can rotate the pieces into the correct orientation. But it is much cheaper and less error-prone to regard the possible orientations of the pieces as states of an automaton on which transitions can be performed by simple machinery, and apply a reset word before the pieces arrive at the robot.

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- Bioinformatics: If a soup of DNA molecules is to perform some computation, we need the molecules to be all in a known state first. We can simultaneously apply a reset word to all of them, where the transitions are induced by some chemical or biological process.

The road-colouring problem

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Suppose that D is a directed graph in which all edges have out-degree d. Then the edges of D can be coloured with d colours to produce an automaton with a reset word if and only if D is strongly connected and the greatest common divisor of the cycle lengths in D is 1.

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This was the road-colouring conjecture until it was proved by Avraham Trahtman last year.

The Černý conjecture

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This is the Černý conjecture, and is still open. If true, the bound would be best possible.

A group-theoretic approach

At the other extreme from a synchronizing automaton is one in which all the transitions are permutations (and generate a permutation group). One approach to the Černý conjecture is to separate out this difficulty.

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Which permutation groups are synchronizing?

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A permutation group G on Ω is non-synchronizing if and only if there is a non-trivial partition π of Ω and a subset Δ of Ω such that, for all $g \in G$, Δg is a section (of transversal) of π.

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Corollary

A synchronizing group is primitive.

For if there is a G-invariant partition π, then any section of π has the required property.

Separating groups

Let G be transitive on Ω, with $|\Omega| = n$. Let Γ and Δ be subsets of Ω, with $|\Gamma| = k$, $|\Delta| = l$.

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Proposition

A separating group is synchronizing.

For if G is non-synchronizing, and Γ is a part of a partition π for which (π, Δ) witness the non-synchronization, then by assumption $|\Gamma \cap \Delta g| = 1$ for all $g \in G$.

Separation and synchronization

Since synchronizing groups are primitive, the obvious first step is to check primitive groups of small degree (up to a few hundred) for these properties. MAGMA and GAP contain lists of these groups. But the checking is non-trivial.

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In particular, we only know a tiny handful of permutation groups which are synchronizing but not separating; it would be interesting to find out why this property is so rare.

Some of the examples come from finite geometry (involving properties of ovoids and spreads in polar spaces), but others appear to be "sporadic".

Graph-theoretic characterisations

These properties can be detected by undirected graphs admitting the group G. The clique number $\omega(X)$ and the independence number $\alpha(X)$ are the cardinalities of the largest complete and null induced subgraphs of X; the chromatic number $\chi(X)$ is the smallest number of colours required to colour the vertices so that adjacent vertices get different colours. Clearly $\omega(X) \leq \chi(X)$, since vertices of a complete subgraph must get different colours.

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Let G be a permutation group on Ω, with $|\Omega| = n$.

- *G is synchronizing if and only if there is a non-trivial G-invariant graph X for which $\omega(X) = \chi(X)$.*
- *Let G be transitive. Then G is separating if and only if there is a non-trivial G-invariant graph X such that $\omega(X) \cdot \alpha(X) = n$.*

An application: homomorphisms and cores

A homomorphism from a graph X to a graph Y is a map from the vertices of X to the vertices of Y which takes edges to edges (we don't care what it does to non-edges).

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Let X be a graph whose automorphism group is transitive on non-edges. Then either the core of X is a complete graph, or X is itself a core.

However, deciding which possibility occurs is hard ...

The hull of a graph

The hull of a graph X is defined as follows:

- $\mathrm{hull}(X)$ has the same vertex set as X;
- $v \sim w$ in $\mathrm{hull}(X)$ if and only if there is no element $f \in \mathrm{End}(X)$ with $v^f = w^f$.

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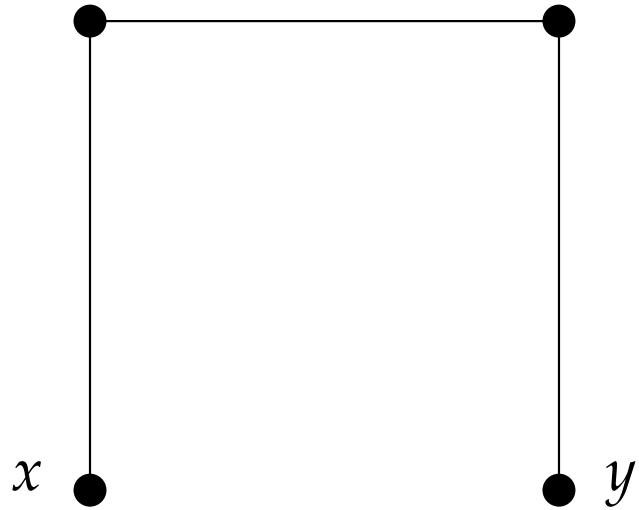
Theorem

- *X is a spanning subgraph of* $\mathrm{hull}(X)$ *(this means that they have the same vertex set, and every edge of X is an edge of* $\mathrm{hull}(X)$*);*
- $\mathrm{End}(X) \leq \mathrm{End}(\mathrm{hull}(X))$ *and* $\mathrm{Aut}(X) \leq \mathrm{Aut}(\mathrm{hull}(X))$ *(End and Aut are the endomorphism semigroup and automorphism group respectively);*
- *if* $\mathrm{core}(X)$ *has m vertices then* $\mathrm{core}(\mathrm{hull}(X))$ *is the complete graph on m vertices.*

An example



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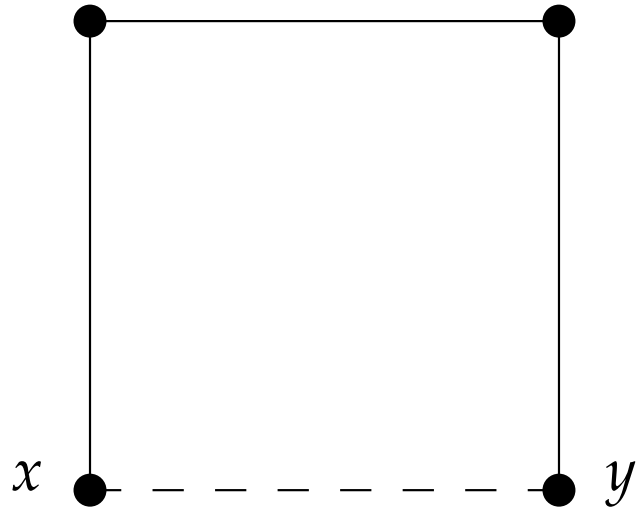

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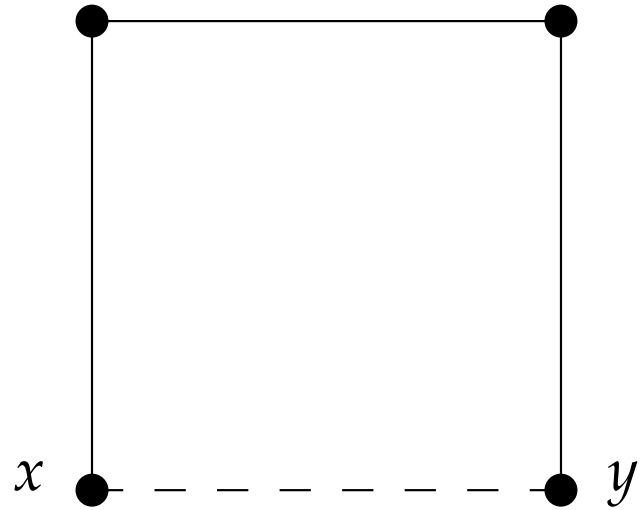

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Note the increase in symmetry: $|\operatorname{Aut}(X)| = 2$ but $|\operatorname{Aut}(\operatorname{hull}(X))| = 8$.

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Let X be non-edge transitive. Then $\mathrm{hull}(X)$ consists of X with some orbits on non-edges changed to edges. So there are only two possibilities:

- $\mathrm{hull}(X) = X$. Then $\mathrm{core}(X) = \mathrm{core}(\mathrm{hull}(X))$ is complete;
- $\mathrm{hull}(X)$ is the complete graph on the vertex set of X. Then $\mathrm{core}(X)$ has as many vertices as X, so that $\mathrm{core}(X) = X$.

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A permutation group is synchronizing (resp. separating) if and only if its 2-closure is synchronizing (resp. separating).

This is because failure of these properties is "detected" by a graph admitting the group (and hence admitting its 2-closure).

More general closure properties

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Proposition

For any field F, a permutation group is synchronizing (resp. separating) if and only if its F-closure is synchronizing (resp. separating).

An example

The group $\mathrm{PSL}(2, 2^n)$ has permutation actions of degrees $2^{n-1}(2^n \pm 1)$, on the cosets of its maximal dihedral subgroups of orders $2(2^n \mp 1)$. It is 2-closed in both actions.

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The permutation character of the action of degree $2^{n-1}(2^n - 1)$ is the sum of the trivial character and a family of algebraically conjugate characters, whose sum is $\mathbb{Q}$-irreducible. So the $\mathbb{Q}$-closure is the symmetric group, which is trivially separating; so the original group is separating, and hence synchronizing. (This was the example of Arnold and Steinberg.)

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More generally, a QI-group (one whose permutation character is the sum of the trivial character and a rational irreducible) is separating.

Another example

We take the same group $\mathrm{PSL}(2, 2^n)$, with $2^n - 1$ a Mersenne prime, and consider the other action described earlier.
The permutation character of the action of degree $2^{n-1}(2^n + 1)$ is equal to the above character plus an irreducible of degree 2^n.
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So again, the original group is separating, and hence synchronizing.