Cores, hulls and synchronization

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Notation

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For a graph Γ, we use $\omega(\Gamma)$ for the clique number, $\chi(\Gamma)$ for the chromatic number, $\overline{\Gamma}$ for the complement, $\alpha(\Gamma)$ for the independence number (so that $\alpha(\Gamma) = \omega(\overline{\Gamma})$), and $\mathrm{Aut}(\Gamma)$ for the automorphism group of Γ.

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Example:

- $K_m \to \Gamma$ if and only if $\omega(\Gamma) \geq m$;
- $\Gamma \to K_m$ if and only if $\chi(\Gamma) \leq m$.

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Thus, the core of Γ is complete if and only if $\omega(\Gamma) = \chi(\Gamma)$.

Proposition

If Γ is vertex-transitive, then so is $\mathrm{core}(\Gamma)$. *Similarly for other kinds of transitivity.*

Rank 3 graphs

A graph Γ is a rank 3 graph if its automorphism group is transitive on vertices, ordered edges and ordered non-edges; in other words, $\mathrm{Aut}(\Gamma)$ is a rank 3 permutation group. (The rank of a permutation group G on a set V is the number of G-orbits on $V \times V$.)

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This is true; the proof came from an unexpected direction: automata theory.

The cave

You are in a dungeon consisting of a number of rooms. Passages are marked with coloured arrows. Each room contains a special door; in one room, the door leads to freedom, but in all the others, to instant death. You have a schematic map of the dungeon, but you do not know where you are.

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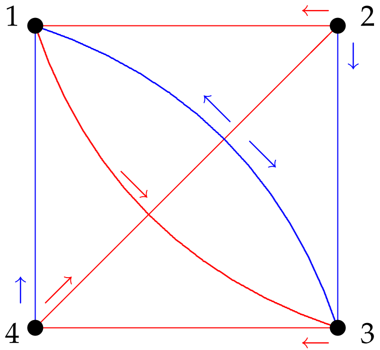

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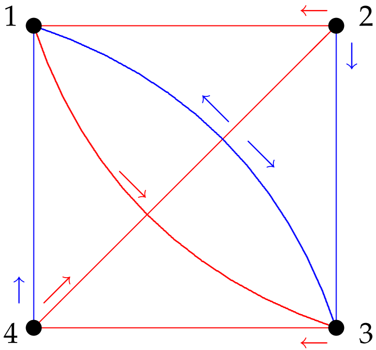

You can check that (Blue, Red, Blue, Blue) is a reset word which takes you to room 3 no matter where you start.

Automata and reset words

An automaton is an edge-coloured digraph with one edge of each colour out of each vertex. Vertices are states, colours are transitions. A reset word is a word in the colours such that following edges of these colours from any starting vertex always brings you to the same state. An automaton which possesses a reset word is called synchronizing.

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Not every finite automaton has a reset word; the Černý conjecture, states that, if a reset word exists, then there is one of length at most $(n-1)^2$, where n is the number of states (or rooms in our example).

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Theorem

A permutation group G on V is non-synchronizing if and only if there is a non-complete and non-null graph Γ on V with $\mathrm{core}(\Gamma)$ complete such that $G \leq \mathrm{Aut}(\Gamma)$.

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Theorem

A permutation group G on V is non-synchronizing if and only if there is a non-complete and non-null graph Γ on V with $\mathrm{core}(\Gamma)$ *complete such that* $G \leq \mathrm{Aut}(\Gamma)$.

Proof.

Let S be a semigroup containing G but no constant function: join v to w if no $f \in S$ satisfies $v^f = w^f$. □

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Theorem

Let Γ *be a nonedge-transitive graph. Then either*

- $\mathrm{core}(\Gamma)$ *is complete, or*
- Γ *is a core.*

The hull of a graph

The hull of a graph Γ is defined as follows:

- $\mathrm{hull}(\Gamma)$ has the same vertex set as Γ;
- $v \sim w$ in $\mathrm{hull}(\Gamma)$ if and only if there is no element $f \in \mathrm{End}(\Gamma)$ with $v^f = w^f$.

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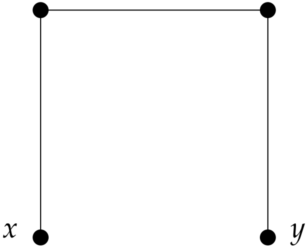
Theorem

- *Γ is a spanning subgraph of* $\mathrm{hull}(\Gamma)$*;*
- $\mathrm{End}(\Gamma) \leq \mathrm{End}(\mathrm{hull}(\Gamma))$ *and* $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\mathrm{hull}(\Gamma))$*;*
- *if* $\mathrm{core}(\Gamma)$ *has m vertices then* $\mathrm{core}(\mathrm{hull}(\Gamma))$ *is the complete graph on m vertices.*

An example



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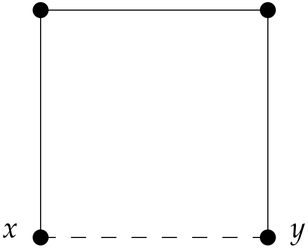

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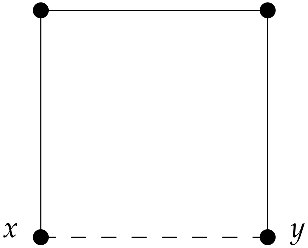

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Note the increase in symmetry: $|\operatorname{Aut}(\Gamma)| = 2$ but $|\operatorname{Aut}(\operatorname{hull}(\Gamma))| = 8$.

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Let Γ be non-edge transitive. Then $\mathrm{hull}(\Gamma)$ consists of Γ with some orbits on non-edges changed to edges. So there are two possibilities:

- $\mathrm{hull}(\Gamma) = \Gamma$. Then $\mathrm{core}(\Gamma) = \mathrm{core}(\mathrm{hull}(\Gamma))$ is complete;
- $\mathrm{hull}(\Gamma)$ is the complete graph on the vertex set of Γ. Then $\mathrm{core}(\Gamma)$ has as many vertices as Γ, so that $\mathrm{core}(\Gamma) = \Gamma$.

Questions about hulls

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If the third question is hard, so are the other two.

Separating permutation groups

Neumann's separation lemma states:

Proposition

Let G be a transitive permutation group on V, with $|V| = n$, and let A, B be subsets of V. If $|A| \cdot |B| < n$, then there exists $g \in G$ with $A^g \cap B = \emptyset$.

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We call a transitive permutation group separating if, for any sets A, B with $|A|, |B| > 1$ and $|A| \cdot |B| = n$, there exists g with $A^g \cap B = \emptyset$.

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None of these implications reverses. (But I have only a single example of a permutation group which is synchronizing but not separating, namely $\mathrm{P}\Omega(5,3)$, acting on 40 points.)

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- *The permutation group G is non-synchronizing if and only if there is a graph Γ (not complete or null) with $\omega(\Gamma) = \chi(\Gamma)$ and $G \leq \mathrm{Aut}(\Gamma)$.*

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- *The permutation group G is non-synchronizing if and only if there is a graph Γ (not complete or null) with $\omega(\Gamma) = \chi(\Gamma)$ and $G \leq \mathrm{Aut}(\Gamma)$.*
- *The transitive permutation group G is non-separating if and only if there is a graph Γ (not complete or null) with $\omega(\Gamma) \cdot \alpha(\Gamma) = |V(\Gamma)|$ and $G \leq \mathrm{Aut}(\Gamma)$.*

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This is because failure of these properties is "detected" by a graph admitting the group (and hence admitting its 2-closure).

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Proposition

For any field F, a permutation group is synchronizing (resp. separating) if and only if its F-closure is synchronizing (resp. separating).

An example

The group $\mathrm{PSL}(2, 2^n)$ has permutation actions of degrees $2^{n-1}(2^n \pm 1)$, on the cosets of its maximal dihedral subgroups of orders $2(2^n \mp 1)$. It is 2-closed in both actions.

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The permutation character of the action of degree $2^{n-1}(2^n - 1)$ is the sum of the trivial character and a family of algebraically conjugate characters, whose sum is $\mathbb{Q}$-irreducible. So the $\mathbb{Q}$-closure is the symmetric group, which is trivially separating; so the original group is separating, and hence synchronizing. (This was the example of Arnold and Steinberg.)

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The permutation character of the action of degree $2^{n-1}(2^n + 1)$ is equal to the above character plus an irreducible of degree 2^n. So its $\mathbb{Q}$-closure is the group S_{2^n+1} acting on 2-sets, which is separating. (The only invariant graphs are the line graph of K_{2^n+1} and its complement; and if $\Gamma = L(K_{2^n+1})$, then $\omega(\Gamma) = 2^n$, but $\alpha(\Gamma) = 2^{n-1}$.) So again, the original group is separating, and hence synchronizing.