Combinatorial Yang–Baxter

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The Yang–Baxter equation

The Yang–Baxter equation (YBE) is a kind of braiding equation for a linear map R on $V \otimes V$.
Define two linear maps on $V \otimes V \otimes V$, namely R^{12}, acting on the first two components of a 3-tensor and fixing the third, and R^{23}, acting on the second and third component and fixing the first.
The Yang–Baxter equation is

$$R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23}.$$

Combinatorial Yang–Baxter

The combinatorial (or set-theoretic) YBE is defined for a map r on $X \times X$, where X is a set.
It asserts, that, as maps from $X \times X \times X$ to itself,

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23},$$

where r^{12} acts as r on the first two coordinates fixing the third, and r^{23} is similarly defined.
Of course, a solution of this equation gives a solution of the "linear" equation over any field F, on setting $V = FX$, the F-vector space with basis X.

Why?

There is a considerable body of work on the combinatorial Yang–Baxter equation; I am not as familiar with it as I should be.

So for me the purpose of this work is: Describing solutions of the combinatorial Yang–Baxter equation is an interesting exercise; it may be useful to get some idea of just how wild they are, and what constraints (if any) they satisfy.

But there is more.

This is by no means a complete survey; I am talking about joint work with Tatiana Gateva-Ivanova, and in particular on my contributions to this work.

What does the CYB equation mean?

We can describe the function r in a different way. If $r(x,y) = (u,v)$, we set $u = f_x(y)$ and $v = g_y(x)$; for each x, the function f_x is a map from X to itself, and similarly for g_y.

To introduce the next result, here is a familiar analogy.

Contrary to my usual habit, I will write maps on the left, and compose from right to left.

Let $\circ$ be a binary operation on X, and define l_x to be the operation of left translation by x: that is, $l_x(y) = x \circ y$. Then the operation $\circ$ is associative if and only if

$$x \circ y = z \Rightarrow l_x l_y = l_z$$

for all $x, y, z \in X$.

The CYB equation has a similar interpretation: it is a kind of "two-dimensional associative law".

Proposition

Let r satisfy the CYB equation on X. If $r(x,y) = (u,v)$, then $f_x f_y = f_u f_v$.

Proof.

For any $z \in X$,

$$(x,y,z) \xrightarrow{r_{23}} (x, f_y(z), ?) \xrightarrow{r_{12}} (f_x(f_y(z)), ?, ?) \xrightarrow{r_{23}} (f_x(f_y(z)), ?, ?),$$

and

$$(x,y,z) \xrightarrow{r_{12}} (u,v,z) \xrightarrow{r_{23}} (u, f_v(z), ?) \xrightarrow{r_{12}} (f_u(f_v(z)), ?, ?).$$

□

Thus solutions to CYB are "2-dimensional semigroups", and of course we want to understand them.

Further conditions

I impose three further conditions on the function r, namely

- r is an involution;
- r fixes pointwise the diagonal of $X \times X$;
- r is non-degenerate (see below).

Non-degeneracy is the requirement that each of the functions f_x and g_y defined earlier is a bijection on X.

I will use the short term solution for a function r that satisfies the combinatorial Yang–Baxter equation and our extra three conditions. Continuing the earlier analogy, solutions are "2-dimensional groups".

What I say is true for finite sets. Some of the results extend to infinite sets.

Remarks

Sometimes we write $f_x(y) = {}^x y$ and $g_y(x) = x^y$. We think of X acting on Y (on the left) and Y acting on X (on the right).

Non-degeneracy means that these actions are by permutation, so we are in the world of permutation groups.

If the non-degeneracy is relaxed, we would be in the world of transformation semigroups instead, which is much less well understood.

Relaxations of the first two conditions have been studied, and some results are known about this.

Two examples

- The function $r(x, y) = (y, x)$ satisfies the Yang–Baxter equation (essentially by the usual proof that the transpositions $(1, 2)$ and $(2, 3)$ in the symmetric group S_3 satisfy the braid relation), and also our three additional conditions. In this case, the functions f_x and g_y are the identity, for all choices of x and y. This is referred to as the trivial solution.
- The function which swaps $(1, 2)$ with $(3, 1)$, $(1, 3)$ with $(2, 1)$, and $(2, 3)$ with $(3, 2)$ (and fixes all diagonal pairs) is a solution on $X = \{1, 2, 3\}$.

Translation to permutation groups

The functions f and g can be regarded as maps from X into the symmetric group on X. It can be shown that a pair of maps $f, g : X \to \mathrm{Sym}(X)$ arise in this way from a solution if and only if they satisfy the following conditions:

- $f_x(x) = x$;
- $f_{f_x(y)} g_y(x) = x$;
- $f_{f_x(y)} f_{g_y(x)} = f_x f_y$.

The second equation shows that g is determined by f, so everything can be expressed in terms of f.

It also shows that the group generated by the maps g_y is contained in the group generated by the maps f_x; by symmetry these groups are equal.

A problem

In our second example earlier, f_1 and g_1 are equal to the transposition $(2, 3)$, while the other functions are the identity. If the solution is trivial then all maps are the identity.

Problem

Is it true that, if f and g satisfy the above conditions and $|X| > 1$, is it true that there exist two points x and y with $f_x = f_y$?

I will say more about this later.

Other algebraic structures

The group $\mathcal{G}(r)$ generated by the permutations f_x for $x \in X$ is the Yang–Baxter permutation group associated with the solution r.

There is another way of producing a group from a solution of the YBE. For any solution r on X, let

$$G(r) = \langle X \mid xy = uv \text{ whenever } r(x,y) = (u,v) \rangle$$

This is the Yang–Baxter group associated with the solution r.

If r is the trivial solution, then $G(r)$ is a free abelian group of rank $|X|$.

One can also consider the semigroup, or the F-algebra (for a field F) generated by X with the same relations.

The algebras are interesting in other contexts: they are quadratic algebras, that is, their relations are homogeneous quadratic expressions in the generators.

Relations

According to Proposition 1, the map $x \mapsto f_x$ extends to a surjective homomorphism from $G(r)$ to $\mathcal{G}(r)$.
Several things are known:

- $\mathcal{G}(r)$ cannot be transitive if $|X| > 1$ (Rump 1989).
- $G(r)$ (and hence $\mathcal{G}(r)$) is a soluble group (Etingov, Schedler and Soloviev 1999).

One of our results is:

Theorem

The derived length of $G(r)$ is one greater than the derived length of $\mathcal{G}(r)$.

Retracts and multipermutation level

Define an equivalence relation called congruence on X by the rule that $x \equiv y$ if $f_x = f_y$.

An affirmative answer to Problem 1 would imply that this equivalence relation is not the relation of equality whenever $|X| > 1$.

It is easy to see that the given solution induces a solution (called a retract) on the set of equivalence classes.

We can repeatedly take retracts; if the corresponding congruences are always non-trivial, eventually we get a solution on a 1-element set. Such a solution is called a multipermutation solution; its level is the number of retractions required to reach the 1-element set.

A multipermutation solution of level 0 is just the trivial solution on a 1-element set. A multipermutation solution of level 1 is the trivial solution on a set of size greater than 1; so the associated permutation group $\mathcal{G}(r)$ is the trivial group, and the group $G(r)$ is (free) abelian.

For a multipermutation solution, the YB group is soluble with derived length bounded by the multipermutation length; so the derived length of the YB permutation group is bounded by one less than the multipermutation length.

Level and cardinality

We have constructions of solutions with finite multipermutation level, which indicate that the cardinality of X grows exponentially with the multipermutation level; our smallest example of level m has cardinality $2^m + 1$.
We also have a structure theorem for finite solutions with abelian YB permutation group. They are necessarily multipermutation, with level at most the number of orbits of $\mathcal{G}(r)$. They can be constructed from the solutions corresponding to the orbits by a construction known as strong twisted union. Every finite abelian group is isomorphic to the YB permutation group of such a solution.
There is surely much more to say about this situation!

Sketch proof of Theorem 2

Let G and G^* be the YB group and permutation group associated with a solution r. We have to show that $\mathrm{dl}(G) = \mathrm{dl}(G^*) + 1$.
First note that we have a homomorphism from G to G^*, mapping x to f_x. From the definition of the group G we find that the kernel of the homomorphism is abelian, so $\mathrm{dl}(G) \leq \mathrm{dl}(G^*) + 1$. Indeed, the kernel is free abelian.

Lemma

Let A be free abelian of finite rank, and H a finite group acting faithfully on A. Then $[H, A] = \langle {}^h a - a : a \in A, h \in H \rangle$ is non-zero, and H acts faithfully on $[H, A]$.

Now we prove the theorem. It is known that there is a natural number p such that the subgroup of G generated by the pth powers of the generators is a free abelian group A, and admits a faithful action by G^*. Define A_n inductively by $A_0 = A$ and $A_{n+1} = [G^{(n)}, A_n]$. Using our lemma, if $G^{(n)} \neq 1$, then $A_{n+1} \neq 1$; so $A_l \neq 1$, where $l = \mathrm{dl}(G^*)$. Since $A_l \leq G^{(l)}$, we see that $\mathrm{dl}(G) > \mathrm{dl}(G^*)$, and so by the inequality at the start of this section, we have $\mathrm{dl}(G) = \mathrm{dl}(G^*) + 1$.

Wreath products

We can construct solutions with arbitrarily large multipermutation level and derived length by the *wreath product* construction.
Let (X_0, r_0) and (Y, r_Y) be solutions. Let X be the disjoint union of $|Y|$ copies of X_0, say $\bigcup_{\alpha \in Y} X_\alpha$, and define a function r on $X \times X$ as follows:

$$\begin{aligned} r(x,x') &= r_\alpha(x,x') && \text{if } x,x' \in X_\alpha \text{ for some } \alpha, \\ r(x,x') &= (x',x) && \text{otherwise.} \end{aligned}$$

Define a map $\sigma : Y \to \mathrm{Sym}(X)$ by the rule that $\sigma(y)$ acts on the copies of X_0 in the same way that f_y acts on Y. Then we can construct a solution on $Z = X \cup Y$: we use the given solution on Y, and the solution just constructed on X, and set $r(y, x) = (\sigma(y)x, y)$ and $r(x, y) = (y, \sigma(y)^{-1}x)$ for $x \in X, y \in Y$.

Proposition

The above construction gives a solution. Its permutation group G^*_Z is the wreath product of $G^*_{X_0}$ and G^*_Y, acting in the usual (imprimitive) way on X and as the top group G^*_Y on Y.
Its multipermutation level is given by
$\mathrm{mpl}(Z) = \mathrm{mpl}(X_0) + \mathrm{mpl}(Y) - 1.$

It is clear that iterating the wreath product on a small solution (such as our three-element solution) will produce solutions whose size grows exponentially with the multipermutation level (or derived length).
The bounds it gives are not best possible; it is possible to bring them down with extra care.

Some problems

- Is it true that the congruence relation defined by any solution on a set of size greater than 1 is not the relation of equality? (This would imply that every finite solution is multi-permutation.)
- Is it true that every finite soluble group $\mathcal{G}$ is a YB permutation group? If so, what is the smallest cardinality of a corresponding solution?
- Is it possible to use the theory of transformation semigroups to extend some of these results to the degenerate case?