Acyclic orientations and poly-Bernoulli numbers

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(with Celia Glass and Robert Schumacher)

This talk reported on a serendipitous discovery that the number of acyclic orientations of a complete bipartite graph is a poly-Bernoulli number, as defined by Masanobu Kaneka in 1997.

1 Acyclic orientations

In these notes, G will be a graph with n vertices and m edges; the vertex and edge sets are V and E respectively. All graphs are simple.

An *acyclic orientation* of a graph is an orientation of the edges of the graph such that there are no directed cycles. I will often abbreviate "acyclic orientation" to a.o.

Any acyclic orientation can be obtained by ordering the vertices and then orienting all edges from the smaller to the greater vertex.

Any acyclic orientation gives rise to a colouring, where the colour given to a vertex is the length of the longest directed path ending at the vertex. Depending on the a.o. chosen, this might be anything from the chromatic number of the graph up to the length of the longest path. It would be interesting to know something about the distribution of the number of colours over all a.o.s of a given graph.

Define the *distance* between two a.o.s of G to be the number of edges oriented differently in the two a.o.s. It is not hard to show that any a.o. has an edge whose orientation can be flipped while preserving the a.o. property. Even stronger, we can transform any a.o. into any other by flipping just those edges which are oriented differently, so the distance is equal to the minimum number of edge flips required to transform one into the other.

The edge flips define a Markov chain on the set of a.o.s of G: choose a random edge and flip it if possible. It would be nice to modify this Markov chain so as to bias it towards a.o.s which give rise to colourings with few colours, but we have not succeeded in doing this.

2 Counting a.o.s

The number of a.o.s of a given graph is an interesting graph parameter. Stanley showed that it is equal to $(-1)^n P_G(-1)$, where P_G is the chromatic polynomial of G. So it is an evaluation of the Tutte polynomial. It is hard to compute exactly, and it is currently unknown whether there is an efficient approximation procedure (fpras) for it.

We'd like to understand the distribution of the number of a.o.s of labelled graphs with n vertices and m edges. The average number can be computed from a theorem of Bender, Richmond, Robinson and Wormald. Let $a(n,m)$ be the number of labelled acyclic digraphs with n vertices and m arcs, and $A_n(x)$ the generating polynomial $\sum_{m=0}^{n(n-1)/2} a(n,m)x^m$. The theorem states:

$$A_n(x) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (1+x)^{i(n-i)} A_{n-i}(x).$$

From this recurrence it is straightforward to calculate $a_{n,m}$. Now the required average is obtained by dividing this number by the number $\binom{n(n-1)/2}{m}$ of labelled graphs with n vertices and m edges. The results (rounded to the nearest integer) for various values of n and m form the middle row of the following table:

The graphs with the minimum number of a.o.s were described by me, Celia Glass and Robert Schumacher. They are obtained, for given n and m, by the following procedure. Make the largest possible complete graph (say K_q) from the given m edges; use the remaining $r = m - q(q-1)/2$ edges to join one further vertex to r vertices of this complete graph. The number of a.o.s of the resulting graph is $q!(r+1)$. Note that this function grows linearly between successive triangular numbers. The numbers form the top row of the table.

The maximum presents much greater difficulties, and we have little proved about this. However, we have a conjecture. A *Turán graph* is a complete

vertices	2	4	6	8	10	12
edges	1	4	9	16	25	36
Min # a.o.s	2	12	96	1440	25200	362880
Ave # a.o.s	2	12	167	3851	156636	9312017
Max # a.o.s?	2	14	230	6902	329462	22934774

Table 1: Numbers of acyclic orientations of graphs with $m = n^2/4$

multipartite graph with the parts of nearly equal sizes (i.e., differing by at most one). In other words, we remove from the complete graph the edges of a collection of disjoint complete graphs of nearly equal sizes covering all the vertices.

Conjecture: If n and m are such that a Turán graph exists, then it maximises the number of a.o.s.

This is known only in the case where the number of edges is so large that the complete graphs have sizes 1 and 2 (i.e., removing at most $n/2$ edges from K_n, the best way is to remove pairwise disjoint edges).

For even n, the complete bipartite graph $K_{n,n}$ is a Turán graph. Rob computed the number of a.o.s for small n, checked in the On-Line Encyclopedia of Integer Sequences, and found the sequence listed in the bottom row of the table. The references led him to papers on poly-Bernoulli numbers.

3 Poly-Bernoulli numbers

This is only a very brief introduction.

Kaneko gave the following definitions. Let

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k},$$

and let

$$\frac{\mathrm{Li}_k(1-\mathrm{e}^{-x})}{1-\mathrm{e}^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

The numbers $B_n^{(k)}$ are the *poly-Bernoulli numbers of order k*.

He gave a couple of nice formulae for the poly-Bernoulli numbers of negative order, of which the following is relevant here:

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 S(n+1, j+1) S(k+1, j+1),$$

where $S(m,p)$ is the *Stirling number of the second kind*, the number of partitions of a set of cardinality m with p parts.

This formula has the (entirely non-obvious) corollary that these numbers have a symmetry property: $B_n^{(-k)} = B_k^{(-n)}$ for all non-negative integers n and k.

4 Connection

We showed that the number of acyclic orientations of the complete bipartite graph K_{n_1,n_2} is the poly-Bernoulli number $B_{n_1}^{(-n_2)}$.

Here is the proof.

Let the bipartition of the vertex set of the graph have parts A and B, which we think of as coloured amber and blue. Any acyclic orientation is defined by a sequence $(A_1, B_1, \ldots, A_k, B_k)$, where the A_i form a partition of A and the B_i of B, where all parts are non-empty except possibly A_1 and B_k. To get round having to consider four cases, we add a new vertex a_0 to A_1 and a new vertex b_0 to B_k. So we get an acyclic orientation by first choosing partitions (which can be done in $S(n_1+1, k)S(n_2+1, k)$ ways), ordering them with A_1 first and B_k last (in $((k-1)!)^2$ ways), and then removing the added vertices a_0 and b_0. Finally, sum over k. The result is

$$\sum_{k=1}^{\min(n_1,n_2)+1} ((k-1)!)^2 S(n_1+1, k) S(n_2+1, k),$$

which is clearly the same as Kaneko's formula.

In a more recent paper, Chad Brewbaker gave another counting interpretation of these numbers. He defines a *lonesum matrix* to be a zero-one matrix which is uniquely determined by its row and column sums. Ryser showed that a binary matrix is lonesum if and only if it does not contain $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as a submatrix (in not necessarily consecutive rows or columns). (If one such submatrix occurred it could be flipped into the other without changing the row and column sums.)

Now the number of $n_1 \times n_2$ lonesum matrices is equal to the number of a.o.s of K_{n_1,n_2}. There is a simple proof of this. Number the vertices in the bipartite blocks from 1 to n_1 (in A) and from 1 to n_2 (in B). Now given an orientation of the graph, we can describe it by a matrix whose (i, j) entry is 1 if the edge from vertex i of A to vertex j of B goes in the direction from A to B, and 0 otherwise. The two forbidden submatrices for lonesum matrices correspond to direceted 4-cycles; so any a.o. gives us a lonesum matrix.

To see the converse, we claim that if an orientation of a complete bipartite graph contains no directed 4-cycles, then it contains no directed cycles at all. For suppose that there are no directed 4-cycles, but there is a directed cycle $(a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_1)$. Then the edge between a_2 and b_2 must be directed from a_1 to b_2, since otherwise there would be a 4-cycle $(a_1, b_1, a_2, b_2, a_1)$. But then we have a shorter directed cycle $(a_1, b_2, a_3, \ldots, b_k, a_1)$. Continuing this shortening process, we would eventually arrive at a directed 4-cycle, a contradiction.

(This says that the cycle space of the complete bipartite graph is generated by 4-cycles.)