Orbital versions of graph parameters

Peter J. Cameron

University of St Andrews / Queen Mary University of London

44th Southeastern Conference on
Combinatorics, Graph Theory and Computing
8 March 2013

"I count a lot of things that there's no need to count," Cameron said. "Just because that's the way I am. But I count all the things that need to be counted."

Richard Brautigan, *The Hawkline Monster: A Gothic Western*

Every graph theorist knows that the colorings of a graph with a given number of colorings are counted by a certain polynomial, the chromatic polynomial of the graph.

But there is more to it. I will describe three problems to which polynomials provide the answer, and a fourth to which the answer is not yet known.

If time permits, I will consider some generalisations where we don't yet know what to do.

Warning: There will be graphs and groups. A typical group will be called G, so a typical graph has to be something else; I will use X.

I will illustrate with the Petersen graph:
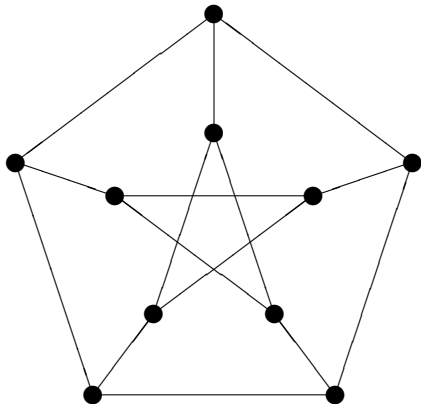
Graph coloring

Graph coloring

A graph, then, is something like the above picture; it has vertices and edges, each edge joining two distinct vertices and not having a direction, and no multiple edges (so each pair of vertices is either joined or not).

A proper coloring of the graph X with q colors is an assignment of the colors to the vertices in such a way that vertices joined by an edge receive different colors.

Theorem
There is a polynomial P_X with the property that, for any positive integer q, the number of proper colorings of X with q colors is $P_X(q)$.

Inclusion–Exclusion

This theorem is usually proved by deletion and contraction.
Here is a proof using Inclusion–Exclusion.
The Principle of Inclusion and Exclusion, or PIE for short,
states:

Theorem
*Let $T_1, \ldots, T_n$ be subsets of a set S. For any subset J of $\{1, \ldots, n\}$, let
T_J be the intersection of the sets T_i for $i \in J$, with $T_\emptyset = S$. Then the
number of elements lying in none of the sets T_i is given by*

$$\sum_{J \subseteq \{1, \ldots, n\}} (-1)^{|J|} |T_J|.$$

The idea is that we count all the elements of $T_\emptyset = S$, remove
the elements that lie in some set T_i, put back in the ones which
have been removed twice, …

Colorings

Counting proper colorings is a job for this principle. We take the set of all possible colorings, proper or not, of the graph with q colors; there are q^n of these, where n is the number of vertices. Now we have to exclude the ones that have bad edges, joining vertices of the same color. How many colorings have at least a set A of bad edges? To count them, look at the subgraph with edge set A. By assumption, all vertices in a connected component of this graph have the same color. So, if $c(A)$ is the number of connected components of this subgraph, there are $q^{c(A)}$ such colorings. Then PIE gives the number of colorings with no bad edges as

$$P_X(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{c(A)}.$$

This is the chromatic polynomial of X.

The Petersen graph

For the Petersen graph, the chromatic polynomial is

$$P_X(q) = q(q-1)(q-2) \times$$
$$(q^7 - 12q^6 + 67q^5 - 230q^4 + 529q^3 - 814q^2 + 775q - 352).$$

We see that the least number of colors required for a proper coloring (the smallest q for which this is non-zero) is 3, and the number of proper colorings for q up to 10 is given in the following table:

q	3	4	5	6	7	8	9	10
$P_X(q)$	120	12960	332880	3868080	27767880	144278400	594347040	2055598560

Variations

Up to order of colors

Often it does not matter which colors are assigned to which vertices; only the partition of vertices into color classes is important. Each color class is an independent set, a set containing no edges; so we want to count partitions into independent sets.

We can't just divide the numbers in the preceding table by q! [WHY?]

First we have to count the colorings in which every color actually appears.

This is once again a job for PIE. Let S be the set of all colorings with q colors, and for each i let T_i be the set of colorings in which color i does not appear. We want to count colorings lying in none of the sets T_i. Defining T_J as in PIE, we see that if $|J| = q - r$, then T_J consists of colorings using just r prescribed colors, so has size $P_X(r)$. So, by PIE, the number of colorings using q colors, all of which actually appear, is

$$P_X^*(q) = \sum_{r \leq q} (-1)^{q-r} \binom{q}{r} P_X(r).$$

(Note that this is *not* a polynomial, since it is zero if $q > n$.) Dividing this number by $q!$, we obtain the number of partitions into q independent sets.

The Petersen graph

For the Petersen graph, the numbers are

q	3	4	5	6	7	8	9	10
$P_X^*(q)/q!$	20	520	2244	2865	1435	315	30	1

Orbit-counting

Return to colorings in the original sense. Perhaps we don't want to count two colorings as different if they agree up to symmetry of the graph, that is, there is a graph automorphism which takes one to the other. So what we are counting are the orbits of the symmetry group on the set of colorings.

To get the answer it does not suffice simply to divide the number of colorings by the number of symmetries. The answer is given by the Orbit-Counting Lemma:

Theorem

The number of orbits of a finite group G acting on a finite set X is obtained by counting, for each element $g \in G$, the number of elements of X which are fixed by g, summing these numbers, and dividing the result by $|G|$.

Colorings up to symmetry

Given an automorphism g of a graph X, how many q-colorings does it fix?

Decompose the permutation g of the vertices into cycles; if a coloring is fixed, then vertices in the same cycle must get the same color. So, if any cycle contains an edge, then the number is zero; otherwise, shrink each cycle to a single vertex and count colorings of the resulting graph X/g, noting that each can be extended uniquely to a coloring of the original graph fixed by g. The result is a polynomial in q called the orbital chromatic polynomial associated with the graph X and group G, denoted by $OP_{X,G}(q)$, whose value at a positive integer q is the number of G-orbits on proper q-colorings of X. Thus we have

$$OP_{X,G}(q) = \frac{1}{|G|} \sum_{g \in G} P_{X/g}(q),$$

with the convention that $P_{X/g} = 0$ if X/g contains a loop.

The Petersen graph

In the above formulation, G can be any group of automorphisms of X, but to answer the earlier question, we take it to be the full automorphism group.

To work this out for the Petersen graph, first we have to understand its automorphisms.

A convenient representation of the Petersen graph is as follows. The vertices can be labelled with the 2-element subsets of $\{1, 2, 3, 4, 5\}$; two vertices are joined if and only if their labels are disjoint. It is clear from this description that the symmetric group S_5 acts on the graph, and in fact this is the full automorphism group.

Now the only automorphisms whose cycles contain no edges are the identity, 2-cycles, and 3-cycles on $\{1, 2, 3, 4, 5\}$. (For example, the cycle $(12, 23, 34, 45, 15)$ of the permutation $(1, 2, 3, 4, 5)$ contains an edge from 12 to 34.) There are 10 2-cycles and 20 3-cycles, and the corresponding graphs X/g are shown:
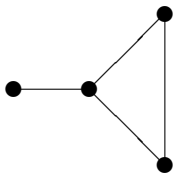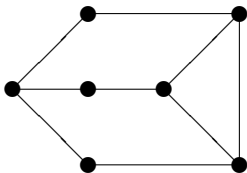
Now doing the calculation, we find that the orbital chromatic polynomial is

$$OP_{X,G}(q) = q(q-1)(q-2) \times$$
$$(q^7 - 12q^6 + 67q^5 - 220q^4 + 469q^3 - 664q^2 + 595q - 252)/120.$$

The values for 3 to 10 colors are

q	3	4	5	6	7	8	9	10
$OP_{X,G}(q)$	6	208	3624	36654	248234	1254120	5089392	17449788

Combining?

What if we don't care about the names of the colors, and also want to count up to symmetry?

The first step of what we did works fine: PIE gives us a formula for the number of orbits on q-colorings in which all the colors are used. The numbers for the Petersen graph are

q	3	4	5	6	7	8	9	10
$OP^*_{X,G}(q)$	6	184	2644	17910	60690	105840	90720	30240

But we cannot simply divide these numbers by $q!$ to get the number of orbits on partitions. This is because it is possible that a permutation of the parts of a partition can be realised by applying a symmetry, so we would be undercounting. Indeed, the numbers in the table are not all divisible by $q!$.

I don't know a mechanical method of finding the number of orbits of G on partitions of X into q independent sets. This would be an interesting research problem. In the case of the Petersen graph, the six orbits with $q = 3$ are indeed all the same if permutations of the colors are allowed, so the first entry in the corresponding table would be 1.

Acyclic orientations

Richard Stanley noticed that, if we substitute -1 into the chromatic polynomial of the graph, we obtain (up to sign) the number of acyclic orientations of the graph, that is, the number of ways of assigning directions to the edges so that no directed cycle is created.

Unfortunately, substituting -1 into the orbital chromatic polynomial doesn't give the number of orbits of G on acyclic orientations of X; but there is another polynomial, the twisted orbital chromatic polynomial, which does this job. It is calculated in the same way as the chromatic polynomial, but terms corresponding to odd permutations g are given a minus sign, that is, subtracted rather than added.

This works because the sign of $P_X(-1)$ is $(-1)^n$, where n is the number of vertices of X. So, if the automorphism g has r cycles, then the sign of the corresponding term in $P_{X,G}(-1)$ is $(-1)^r$, and the sign of the permutation g is $(-1)^{n-r}$; so if we multiply the contribution of each permutation by its sign, the terms add instead of cancelling.

The Petersen graph

The twisted orbital chromatic polynomial of the Petersen graph is given by

$$P^*_{X,G}(q) = q(q-1)(q-2)(q-3) \times$$
$$(q^6 - 9q^5 + 40q^4 - 120q^3 + 229q^2 - 277q + 164)/120.$$

We find that the 16680 acyclic orientations of the Petersen graph fall into 168 orbits under the automorphism group. Note that the twisted orbital chromatic polynomial, like the ordinary chromatic polynomial but unlike the untwisted orbital version, has no negative real roots.

A problem

A great deal is known about chromatic roots, or roots of chromatic polynomials of graphs:

- there are no negative real chromatic roots, no roots in $(0, 1)$, and none in $(1, \frac{32}{27})$, but they are dense in $[\frac{32}{27}, \infty)$ (Bill Jackson, Carsten Thomassen);

- chromatic roots are dense in $\mathbb{C}$ (Alan Sokal).

Problem
What can be said about orbital chromatic roots (ordinary or twisted)?

Another problem

The $\alpha + n$ conjecture asserts that, for any algebraic integer α, there exists a natural number n such that $\alpha + n$ is a chromatic root (and hence so is $\alpha + m$ for all $m \geq n$).

This is true for quadratic integers (easy) and for cubic integers (by Adam Bohn), but is unknown beyond that.

Problem

Investigate algebraic properties of orbital chromatic roots along similar lines.

Generalizations

Generalizations

The chromatic polynomial of a graph is generalized by the two-variable Tutte polynomial, which has specialisations counting many things other than colorings.

The Orbit-Counting Lemma can be systematized by the multivariable cycle index of a permutation group, which yields counts for orbits of the group on very general configurations.

Problem

Combine the two approaches, to count orbits of G on graph-theoretic objects counted by the Tutte polynomial.

I will briefly describe the situation for flows and tensions on a graph. This is taken from a paper with Bill Jackson and Jason Rudd.

Tensions and flows

Let A be a finite abelian group. Choose a fixed but arbitrary orientation of the edges of the graph X.

- A tension on X (over A) is a function from the set of arcs of X to A with the property that the signed sum of the values along any circuit is zero.

- A flow on X (over A) is a function from the set of arcs of X to A with the property that the signed sum of the values on the arcs through any vertex is zero.

We are interested in the numbers of nowhere-zero tensions and flows.
Cleary, reversing an arc and changing the sign of the function there does not affect the property of being a tension or flow; so the numbers of these are independent of the orientation.

Tensions

It is easy to see that any tension is obtained as follows. Choose a function Φ from the vertex set of X to A; now put on each arc the difference between its values at the head and the tail.

Now Φ is a proper coloring of X if and only if the derived tension is nowhere-zero.

There is one free choice for the value of Φ in each connected component. So the number of nowhere-zero flows is $P_X(q)/q^c$, where c is the number of connected components of X, and $q = |A|$. Note that the number does not depend on the structure of A, only its order. We call $P_X(q)/q^c$ the tension polynomial of X.

Flows

It is often claimed that flows are "dual" to colorings. What they are really dual to is tensions, as we will see.

Tutte showed that the number of nowhere-zero flows does not depend on the structure of A, but only on its order; this number is a polynomial in $q = |A|$, called the flow polynomial of the graph X.

For planar graphs, the flow polynomial of a graph is the tension polynomial of its dual.

There are many interesting unsolved problems about roots of flow polynomials.

Orbital versions

There are orbital tension and flow polynomials associated with a graph X and group G of automorphisms of X.

However, unlike the usual polynomials, they are multivariate. We have variables x_i for $i = 0$ and for each positive integer i for which there is an element of order i in the group G.

To obtain a count for the number of orbits of G on tensions and/or flows over an abelian group A, we substitute α_i for x_i in the appropriate polynomial, where α_i is the number of solutions of $ia = 0$ for $a \in A$. Note that $\alpha_0 = |A|$ and $\alpha_1 = 1$; so if G is trivial, the result depends only on $|A|$, in agreement with Tutte's observation.

An example

The Petersen graph is a bit big, so I will consider the following graph. Edges are directed top-to-bottom, and the letters indicate the values (in an abelian group A) of the function.
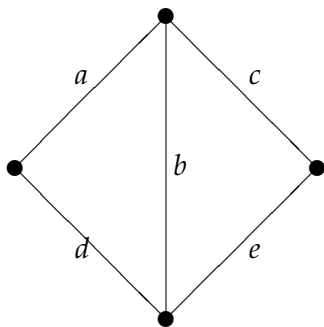
The group G consists of the identity, the left-to-right reflection r_1, the top-to-bottom reflection r_2, and their product.

Orbits on nowhere-zero tensions

Let $q = |A|$ and α_i as before.

- For a tension, we have $a + d = b = c + e$, so there are $(q-1)(q-2)^2$ non-zero tensions.
- A tension is fixed by r_1 if $a = c$. So there are $(q-1)(q-2)$ such tensions.
- A tension is fixed by r_2 if $d = -a$, $e = -c$ and hence $b = 0$. So there are no fixed tensions.
- A tension is fixed by $r_1 r_2$ if and only if $e = -a$, $d = -c$, and $b = -b$. So $2b = 0$ (α_2 choices), then there are $q - 2$ choices for a, after which everything is determined. So $\alpha_2(q-1)$ fixed tensions.

So the orbital tension polynomial is

$$\frac{1}{4}(x_0 - 1)((x_0 - 1)(x_0 - 2) + x_2).$$

Orbits on nowhere-zero flows

- For a flow, we have $d = a$, $e = c$, and $a + b + c = 0$. So $(q-1)(q-2)$ choices.
- A flow is fixed by r_1 if $a = b$. So $c = -2a$ and we require $2a \neq 0$: $q - \alpha_2$ choices.
- A flow is fixed by r_2 if $2a = 2b = 2c = 0$. So $(\alpha_2 - 1)(\alpha_2 - 2)$ choices.
- A flow is fixed by $r_1 r_2$ if $a + e = c + d = 2b = 0$. Then $a + c = 0$, so $b = 0$, and there are no fixed flows.

So the orbital flow polynomial is

$$\frac{1}{4}((x_0 - 1)(x_0 - 2) + (x_0 - x_2) + (x_2 - 1)(x_2 - 2)).$$

Invariant factors and duality, 1

Let R be a principal ideal domain. Given an $m \times n$ matrix M over R, we define the row space of $\rho(M)$ and the null space $\nu(M)$ as usual:

$$
\begin{aligned}
\rho(M) &= \{yM : y \in R^m\}, \\
\nu(M) &= \{x \in R^n : Mx^\top = 0\}.
\end{aligned}
$$

M can be put into *Smith normal form* by elementary row and column operations: this is a matrix with r non-zero diagonal elements $d_1, \ldots, d_r$ and all other entries zero, where d_i divides d_{i+1} for $i = 1, \ldots, r - 1$. The elements $d_1, \ldots, d_r$ are uniquely determined up to multiplication by units of R. They are the *invariant factors* of M. By convention, we also take 0 to be an invariant factor with multiplicity $n - r$, so that there are n invariant factors in all.

Invariant factors and duality, 2

Two matrices M and M^* over the PID R are dual if the row space of M is equal to the null space of M^* and *vice versa*.

A matrix is *totally unimodular* if every subdeterminant is zero or a unit. (This property is *not* preserved by elementary operations.)

Theorem
Let M be a matrix over R. Then the following are equivalent:

- ▶ *M has a dual;*
- ▶ *all invariant factors of M are zero or units;*
- ▶ *M is equivalent (by elementary row and column operations) to a totally unimodular matrix.*

If Γ is a graph with oriented edges, and M and M^* are its signed vertex-edge and cycle-edge incidence matrices, then M and M^* are dual.

Orbital Tutte polynomial, 1

Assume that (M, M^*) is a dual pair over a principal ideal domain R. The linearly independent sets of columns of M are the independent sets of a matroid. The linearly independent sets of columns of M^* form the dual matroid.

An automorphism of M to be an automorphism of the free module R^n (where n is the number of columns of M) which preserves the row space and null space of M.

If g is an automorphism of M (represented as an $n \times n$ matrix), and 1 is the identity matrix, set

$$M_g = \begin{pmatrix} M \\ g-1 \end{pmatrix}, \qquad M_g^* = \begin{pmatrix} M^* \\ g-1 \end{pmatrix}.$$

For any subset S of $E = \{1, \ldots, n\}$, and any matrix N with n columns, we let $N[S]$ be the submatrix of N consisting of the columns with indices in S.

Take two sets $(x_i : i \in I)$ and $(x_i^* : i \in I)$ of indeterminates, where the index set I is the set of associate classes in R. For any matrix N, let $x(N)$ be the monomial defined as follows: take the invariant factors of N (completed with zeros so that the number of them is equal to the number of columns of N), and multiply the corresponding indeterminates. Define $x^*(N)$ similarly, using the other set of indeterminates.

Now let G be a finite group of automorphisms of M, and define the orbital Tutte polynomial $OT(M, G)$ in the indeterminates $(x_i, x_i^* : i \in I)$ as follows:

$$OT(M, G) = \frac{1}{|G|} \sum_{g \in G} \sum_{S \subseteq E} x(M_g[S]) x^*(M_g^*[E \setminus S]).$$

Specialisations

Theorem
If G is the trivial group, then $OT(M, G)$ involves only x_0, x_1, x_0^* and x_1^*; the substitution $x_1 = x_1^* = 1$, $x_0 = y - 1$, $x_0^* = x - 1$ gives the Tutte polynomial of M.

Theorem
If M and M^* are the vertex-edge and cycle-edge incidence matrices of a graph X, and G a group of automorphisms of X, then the substitution $x_i = \alpha_i(A)$, $x_i^* = -1$ (for all i) in $OT(X, G)$ gives the number of G-orbits on nowhere-zero A-flows on X, while the substitution $x_i = -1$, $x_i^* = \alpha_i(A)$ gives the number of G-orbits on nowhere-zero A-tensions on X.