Parties, permutations and diagrams

Peter J. Cameron

University of St Andrews

Flanders Mathematical Olympiad Prizegiving
Leuven, 21 May 2014

The party problem

> *Six people are at a party. Show that either there are three of them, any two of whom know each other, or there are three people, no two of whom know each other.*

The phrase "Show that ..." shows that we are being asked to do some mathematical reasoning. So the first thing we need to do is to be more precise about our terms.

Using the mathematician's prerogative to use words with any defined meaning, we will abbreviate "any two know each other" to "mutual friends", and "no two know each other" to "mutual strangers". We assume also that friendship is an irreflexive symmetric relation: that is,

- nobody is his/her own friend; and
- if A is B's friend, then B is A's friend.

Thus friendship is represented by a subset of the set of 2-element subsets, and "strangership" is the complementary subset. We can represent this in a diagram by drawing a red edge between two friends, and a blue edge between two strangers.

Five people do not suffice

First, observe that five people do not suffice for the assertion of the party problem. For they might form the configuration shown:
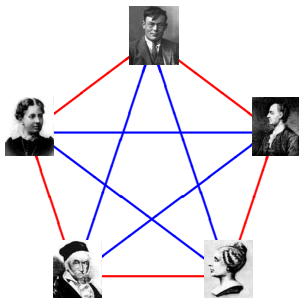
There are no three mutual friends and no three mutual strangers.

Six people do suffice

So here is the proof that six people are enough for the statement to be true. I will call the people A, B, C, D, E, F, and talk about "red edges" and "blue edges" rather than "friends" and "strangers".

Consider person A. Either the number of red edges containing A is at least three, or the number of blue edges containing A is at least three. (For otherwise A is in at most two red and two blue edges, so at most four edges altogether, which is not so.)

Suppose there are three red edges, which might as well be AB, AC and AD. If any two of B, C, D are joined by red edges (say BC), we have a red triangle ABC. But if none of them are, then we have a blue triangle BCD.

The case of three blue edges is similar; just reverse the colours.

The party problem generalised

It is natural to ask: *Is there a minimum number of people at a party which guarantees at least four mutual friends or four mutual strangers*? Indeed there is, and the required number is 18, though this is a bit harder to prove.

What about five or more? The answer is, *we don't know*: that is, we know that the required number always exists, but we don't know what it is.

It is known, for example, that the number for five mutual friends or strangers is between 43 and 49 (that is, 49 people suffice but 42 do not), and for six it is between 102 and 165. The gaps get bigger as the required number of friends or strangers increases.

The view of Paul Erdős

Paul Erdős, the most prolific mathematician of the twentieth century and one of the founders of this theory, regarded this problem as one of the most difficult. He said:

Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Ramsey's Theorem

The party problem is a special case of *Ramsey's Theorem*: not only the size of the subset we look for but also the size of the sets being coloured and the number of colours are arbitrary.

Theorem
Let k, l, r be positive integers with $k \leq l$. Then there is a number n with the following property:

> *Given a set X with n points, if we assign colours $c_1, \ldots, c_r$ arbitrarily to the k-element subsets of X, then there will exist an l-element subset, all of whose k-element subsets have the same colour.*

The smallest number n with this property is the Ramsey number $R(k, l, r)$. So we saw that $R(2, 3, 2) = 6$ and $R(2, 4, 2) = 18$.

The Pigeonhole Principle

There is one special case where we know everything: the case $k = 1$. In this case, colouring the single elements with r colours is equivalent to putting them into r pigeonholes. If we have n pigeons altogether, with $n \geq r(l-1) + 1$, then some pigeonhole must contain at least l pigeons; whereas $r(l-1)$ would not suffice since we could put $l-1$ in each pigeonhole. So $R(1, l, r) = r(l-1) + 1$.

In Scotland, pigeons are called "doos" and they live in a "doocot":

Approximation

Here is an application of the pigeonhole principle.

Theorem
Any irrational number α can be approximated to order 2 *by rational numbers; this means that there are infinitely many rational numbers p/q for which*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

It is known that some irrational numbers, such as the golden ratio $(1 + \sqrt{5})/2$, cannot be approximated to any order higher than 2. (The best rational approximations to the golden ratio are the quotients of successive Fibonacci numbers.)
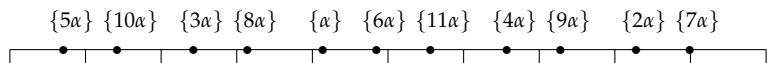
Proof

We show that, given any number n, we can find p/q with $q > n$ such that $|\alpha - p/q| < 1/(nq)$. To see this, we let $\{x\}$ denote the fractional part of the number x, that is, x minus the integer immediately below.

Divide the unit interval into n equal subintervals. Now consider the rational numbers $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots, \{(n+1)\alpha\}$. Since there are $n+1$ numbers in n intervals, there must be two of them in the same interval: so $|\{i\alpha\} - \{j\alpha\}| < 1/n$. Putting $q = |i - j|$, we have $|q\alpha - p| < 1/n$ for some integer p, so that $|\alpha - p/q| < 1/(nq)$. Since $q < n$, we have $|\alpha - p/q| < 1/q^2$.

Why are there infinitely many such numbers? If we have found finitely many, we can choose n large enough that $1/n$ is smaller than $|q'\alpha - p'|$ for any of the fractions p'/q' found so far, and so the p/q we obtain is a new fraction; we can go on as long as we like!

An example

This is what happens if we take $\alpha = \sqrt{2}$ and $n = 10$.

$\{5\alpha\}$ $\{10\alpha\}$ $\{3\alpha\}$ $\{8\alpha\}$ $\{\alpha\}$ $\{6\alpha\}$ $\{11\alpha\}$ $\{4\alpha\}$ $\{9\alpha\}$ $\{2\alpha\}$ $\{7\alpha\}$

We see that the fractional parts of $\sqrt{2}$ and $6\sqrt{2}$ are within $1/10$; so $5\sqrt{2}$ differs from the nearest integer 7 by less than $1/10$, that is, $|\sqrt{2} - 7/5| < 1/50$.

Ramsey theory

Ramsey, who was a logician and mathematical economist, published his theorem in 1930 in the context of mathematical logic. He died in the same year. His younger brother later became the Archbishop of Canterbury.

In the early 1930s, a group of young Hungarian mathematicians including Paul Erdős, George Szekeres, Esther Klein, and Paul Turán rediscovered Ramsey's theorem, but developed it into a much wider theory, which applies not just to sets and subsets but to structures of any kind. This is now referred to as Ramsey theory.

The slogan for Ramsey theory, coined by Theodore Motzkin, is "Complete disorder is impossible": in any structure, no matter how disordered, one can find small patches of order (such as groups of mutual friends in the party problem).

Complete disorder is impossible

The stars are scattered roughly randomly on the sky. But when we look at the sky, our attention is drawn to small groups, such as the three stars in Orion's belt (Alnitak, Mintaka and Alnilam) which appear to form a line, although at vastly different distances from us.

The line also appears to pass through the bright stars Sirius (in Canis Major) and Aldebaran (in Taurus).

Generalization

One advantage of generalization in mathematics is that sometimes the more general problem turns out to be easier than the special case.

Even if this doesn't happen, it may be that the generalization suggests other special cases which can be solved.

One of these is a theorem of Erdős and Szekeres about permutations, to which we now turn.
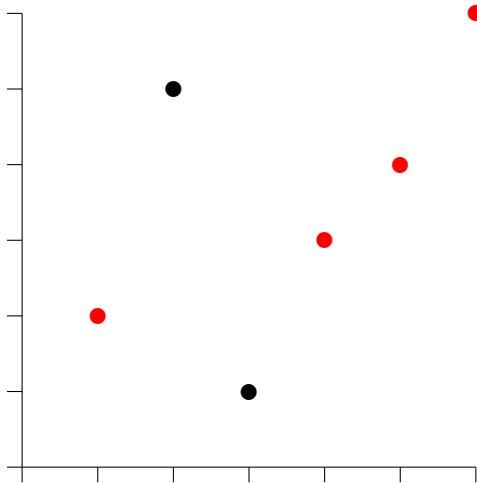
Permutations

There are several ways of thinking of permutations. Today it is most common to say that a permutation is a bijective function from a set to itself; we compose permutations (as functions) and they form a group (the symmetric group).

But to Galois (arguably the inventor of group theory) in 1830, a bijective function was a substitution; a permutation is the resulting re-ordering of the set being permuted, which we usually take to be the set $\{1, 2, \ldots, n\}$.

For example, $(2, 5, 1, 3, 4, 6)$ is a permutation of $(1, 2, 3, 4, 5, 6)$; the corresponding substitution maps 1 to 2, 2 to 5, 3 to 1, and so on.

We can represent permutations graphically by plotting the points (x, y) where y is the number standing in position x in the permutation.

For example, the permutation $(2, 5, 1, 3, 4, 6)$ is shown in this picture:

The Erdős–Szekeres Theoreom

In the permutation $(2, 5, 1, 3, 4, 6)$, the numbers $2, 3, 4, 6$ form an increasing subsequence of length 4 (shown in red in the diagram). There is no decreasing subsequence of length greater than 2. Does any permutation contain a long increasing or decreasing sequence?

Theorem (Erdős–Szekeres)

Given a positive integer k, there exists n such that any permutation of $\{1, \ldots, n\}$ contains either an increasing or decreasing sequence of length k. The smallest k for which this is true is $n = (k-1)^2 + 1$.

I will show you how this follows from the party problem – but this approach is not going to yield an exact value for *n* – and then I will show you a different proof which gives the exact result.

First proof

There are two kinds of pairs of elements in a permutation p, increasing and decreasing. Colour the edge $\{i, j\}$ (with $i < j$) red if $p(i) < p(j)$ (so the pair $p(i), p(j)$ is increasing) and blue if $p(i) > p(j)$ (so the pair $p(i), p(j)$ is decreasing).
A set of k points with all edges red is an increasing subsequence; a set of k with all edges blue is a decreasing subsequence.
By Ramsey's Theorem, if $n \geq R(2, k, 2)$ (the number of people required for a party to have at least k mutual friends or at least k mutual strangers), there is an increasing or decreasing subsequence of length k.
So for $k = 3, 4$ the theorem shows that $n = 6$, resp. $n = 18$, points suffice. But the tailpiece to the theorem tells us that the best possible values should be $n = 5$, resp. $n = 10$.

Second proof

The first proof reduced the problem to a known, but difficult, result. The second attacks it directly.

Suppose that $n = (k-1)^2 + 1$, and we are given a permutation p of $\{1, \ldots, n\}$ which contains no increasing sequence of length k. Now we put the points into $k-1$ pigeonholes $P_1, P_2, \ldots, P_{k-1}$ by the following rule: $i \in P_l$ if the longest increasing sequence which *ends* in position i contains l terms.

By the Doocot Principle, one of the pigeonholes, say P_m, contains at least k numbers. Suppose that $i_1, \ldots, i_k \in P_m$, where $i_1 < \cdots < i_k$.

Now $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ is a *decreasing* sequence of length k. For suppose that, say, $x_{i_1} < x_{i_2}$. There is an increasing sequence of length m ending with x_{i_1} in position i_1; adding x_{i_2} to the end gives a sequence of length $m+1$, which contradicts the fact that the longest increasing sequence ending at x_{i_2} is also m.

Here is the argument in pictorial form:
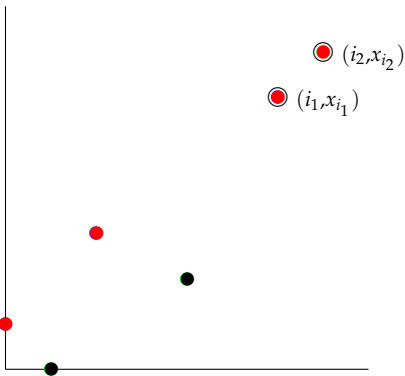
We put i_1 and i_2 in pigeonhole P_3 because they are the ends of increasing sequences of length 3. But if $x_{i_1} < x_{i_2}$ then i_2 should have gone into pigeonhole P_4.

Why $(k-1)^2 + 1$ is best possible

Here is an example with $n = 9$ containing no increasing or decreasing sequence of length 4: $(7, 8, 9, 4, 5, 6, 1, 2, 3)$. I hope you can see how to generalize this to any number k, to show that $(k-1)^2$ is not enough.
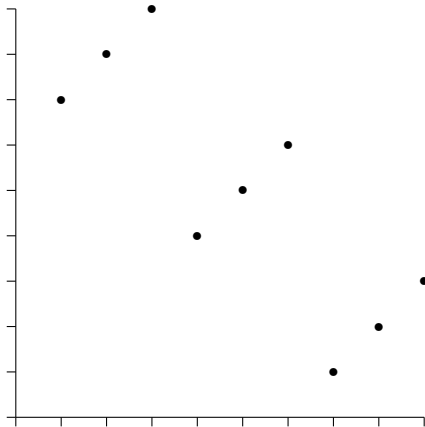
Random permutations

If a permutation has an increasing sequence of length k, then the permutation obtained by reading it in reverse has a decreasing sequence of length k. Since every permutation has an increasing or decreasing sequence of length at least $1 + \sqrt{n-1}$, we see that at least half of all permutations have an increasing sequence of this length.

So the average length of the longest increasing sequence in a random permutation is at least $(1 + \sqrt{n-1})/2$.

The value given by this simple argument is within a constant factor of the correct result. It has been shown by Baik, Deift and Johansson that the average length is asymptotically $2\sqrt{n}$. This leads rapidly into deep mathematical waters, where we meet interesting creatures such as the Tracy–Widom distribution and largest eigenvalue of random unitary matrices.

Deep or not, I am going to tell you a little bit more of this fascinating story …

Young diagrams and tableaux

A **Young diagram** is a collection of n boxes arranged into rows, aligned on the left, so that the lengths of the rows are non-increasing.

A **Young tableau** is a Young diagram with the numbers $1, \ldots, n$ written into the boxes so that the numbers in any row or column increase.

1	2	4
3	6	
5	7	
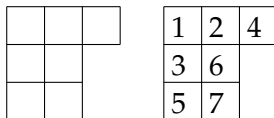
Let f_λ be the number of Young tableaux with shape λ, that is, the number of ways of putting in the numbers $1, \ldots, n$ so that the rows and columns increase. A remarkable relation holds:

$$\sum_\lambda f_\lambda^2 = n!.$$

Here the sum is over all Young diagrams λ.

The Robinson–Schensted algorithm

The formula $\sum f_\lambda^2 = n!$ says that the number of *pairs of tableaux with the same shape* is equal to the number of *permutations*.
The proof involves taking a permutation and "decoding" it into a pair of tableaux of the same shape. It is straightforward but a bit time-consuming to explain in detail, so I will do an example. We start with the permutation $(2, 3, 1)$, and read its elements one at a time.
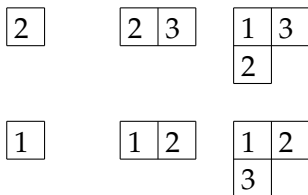
At the third stage, the number 1 "bumps" 2 down to the second row. The second tableau records the appearance of the boxes.

Consequences

The Robinson–Schensted algorithm has a couple of simple but profound consequences:

- The longest increasing sequence in a permutation is equal to the length of the first row of the corresponding Young diagram (and the algorithm gives an efficient way to find this number for a given permutation).
- The inverse permutation produces the same two Young tableaux in the opposite order. This means that a pair consisting of two identical tableaux corresponds to a permutation which is equal to its inverse; so the number of self-inverse permutations is

$$\sum_{\lambda} f_{\lambda}.$$

Random permutations and random diagrams

Since $\sum f_\lambda^2 = n!$, we see that we can define the probability of a Young diagram λ to be $f_\lambda^2 / n!$; these probabilities then add up to 1. (This is known as the Plancherel measure.)

It follows from our argument that choosing a random diagram according to this measure is equivalent to choosing a random permutation (uniformly), applying the Robinson–Schensted algorithm to it, and taking the shape of the resulting diagrams. So we see that the length of the first row in a random Young diagram (according to this procedure) is about $2\sqrt{n}$.

Asymptotics

It has been shown, independently by Kerov and Vershik and by Logan and Shepp, that, in the asymptotic limit of large n, the random Young diagram has a characteristic shape (shown here rotated $45°$ for convenience):