Infinity: from the Greeks to the geeks

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Infinity

What is infinity? Is it

- the counting numbers 1, 2, 3, . . . , going on without end?
- the universe, so big that we can't see the edges?
- . . . or because it is logically impossible for it to have edges?
- something where you can travel forever, even if you return to your starting point, like the symbol for infinity:

$$\infty$$

Infinity

Is infinity a threat? or a promise? or just pure pleasure?

Infinity

Humans have always thought and worried about infinity.
I believe that one of the hardest lessons we learn as children is that we are not immortal, eternal beings: there was a time before we were, and (even harder to bear) there will be a time after we have ceased to be.
It is very difficult to face this fact unflinchingly.

Māluṅkyaputta's questions

Māluṅkyaputta, a disciple of the Buddha, went to his master with a number of questions, to which he desperately wanted answers:

- whether the world is eternal or not eternal,
- whether the world is finite or not,
- whether the soul (life) is the same as the body, or whether the soul is one thing and the body another,
- whether a Buddha (Tathagata) exists after death or does not exist after death, whether a Buddha both exists and does not exist after death, and whether a Buddha is non-existent and not non-existent after death.

The Buddha's answer

The Buddha explained that, whether or not these questions were important, they were less important to a person than various others which the Buddha had talked about:
"Whether the view is held that the world is eternal, or that the world is not eternal, there is still re-birth, there is old age, there is death, and grief, lamentation, suffering, sorrow, and despair."
That may be so, and it may have eased Māluṅkyaputta's doubts, but it has not stopped the quest – and it is not the end of my lectures, since I have to say something about the first two questions!

Infinity on trial

Inside the museums, infinity goes up on trial.
Voices echo, "This is what salvation must be like, after a while."

Bob Dylan, "Visions of Johanna".

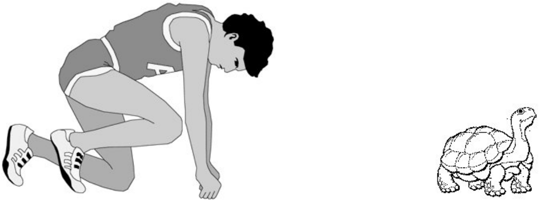
Why had not the Public Prosecutor asked him: "Defendant Rubashov, what about the infinite?" He would not have been able to answer – and there, there lay the real source of his guilt … Could there be a greater?

Arthur Koestler, *Darkness at Noon*

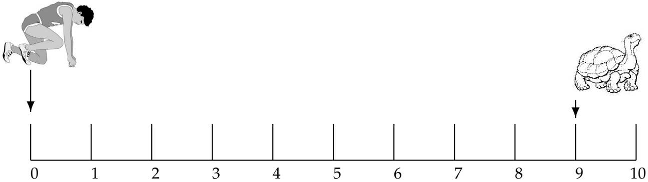
Aristotle

The Greeks had no uniform view on these questions. But the two Greek philosophers who had the greatest effect on subsequent ages were certainly Plato and Aristotle. Aristotle's view was clear. He distinguished between potential infinity and actual infinity. The former refers to a process (like the succession of natural numbers, or successive bisections of an interval) which has no end but is never complete. An actual infinity would be the result of the completion of such a process. Aristotle said that potential infinity was acceptable, but actual infinity was not since it led to paradoxes such as that of Zeno. His influential views put a damper on speculation in Europe for millennia.

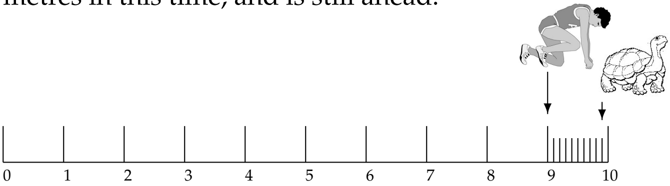
Zeno's paradox



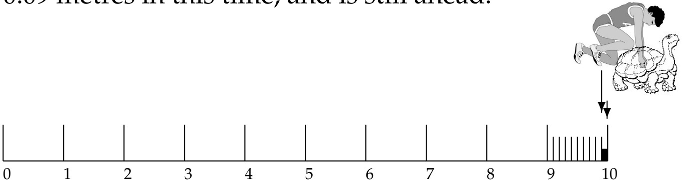
Achilles has a race with a tortoise. Because Achilles runs ten times as fast as the tortoise, he agrees to give it a head start of, say, nine metres. The two contestants start at the same moment. Does Achilles catch the tortoise?



After 0.9 seconds, Achilles has run 9 metres and reached the point from which the tortoise started. But the tortoise has crawled 0.9 metres in this time, and is still ahead.



After another 0.09 seconds, Achilles has run 0.9 metres and again reached the point where the tortoise was. But the tortoise has crawled 0.09 metres in this time, and is still ahead.



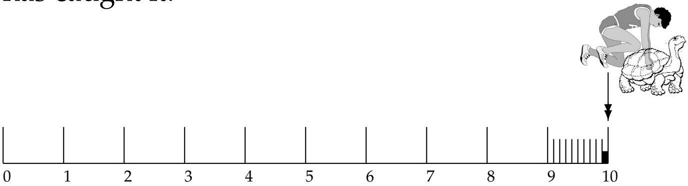
After another 0.009 seconds, Achilles has run 0.09 metres and again reached the point where the tortoise was. But the tortoise has crawled 0.009 metres in this time, and is still ahead.

And so on …

Achilles catches the tortoise

The conclusion is that Achilles never catches the tortoise, since when he reaches the last point the tortoise was, the tortoise has moved on a bit.

Most people nowadays don't see the problem. After one second, Achilles has run ten metres and the tortoise one, so he has caught it.



The problem is to explain what is wrong with the reasoning on the preceding slide.

Angels and pinheads

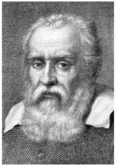
In the Middle Ages the problem of infinity was of interest mainly in connection with arguments about whether the set of angels who could sit on the head of a pin was infinite or not.

N. Ya. Vilenkin, *Stories about Sets*, Academic Press, New York, 1968.

It is now generally believed that this is not the case; the story was put about in the Reformation to discredit the discussions of the mediaeval scholastics.
The earliest known reference seems to be in William Chillingworth's *The Religion of Protestants*, 1637.
But 20th century physics has given this question new importance . . .

Galileo

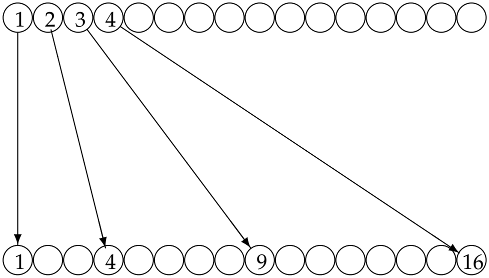
Galileo discussed infinity in his *Dialogue concerning two new sciences*. Let's look at his argument.



Salviati: I take it for granted that you know which of the numbers are squares and which are not.
Simplicio: I am quite aware that a squared number is one which results from the multiplication of another number by itself; thus 4, 9, etc., are squared numbers which come from multiplying 2, 3, etc., by themselves.
Salv: Very well; and you also know that just as the products are called squares so the factors are called sides or roots; while on the other hand those numbers which do not consist of two equal factors are not squares. Therefore if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not?

Simp: Most certainly.
Salv: If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.
Simp: Precisely so.
Salv: But if I inquire how many roots there are, it cannot be denied that there are as many as the numbers because every number is the root of some square. This being granted, we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots.

Here is a diagram of Galileo’s argument. The squares can be matched up with the numbers; but clearly the squares form only a very thin subset of all the numbers.



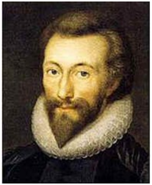
Each number can be matched with its square; yet the squares “thin out” more and more the further we go along.

Sagredo: What then must one conclude under these circumstances?
Salv: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes “equal”, “greater”, and “less”, are not applicable to infinite, but only to finite, quantities.

So Galileo, after a thoroughly modern analysis of the problem, draws back and concludes that infinity is a number, but such numbers cannot be compared or used for arithmetic.

What is infinity plus one?

The metaphysical poet John Donne, in his poem "Loves Growth", was perplexed by the question whether it is possible to make an infinite set bigger by adding something to it:



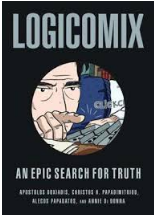
Methinks I lied all winter, when I swore
My love was infinite, if spring make it more.

Good question. If you add one to a number, you get a bigger number. So presumably either "infinity plus one" is bigger than infinity, or infinity is not a number.
I will talk about this in the third lecture.

A dangerous idea

A few years ago, I contributed to a BBC World Service radio programme about infinity. My contribution was boiled down to a few soundbites. After the last of these, psychiatrist Raj Persaud was interviewed, and explained how in his clinic he saw many people who had gone mad thinking about infinity (clearly suggesting that the crazy mathematician who had just been talking would be the next).

One of the most successful graphic novels of the last few years was *Logicomix*, by Apostolos Doxiadis and Christos Papadimitriou. (If you haven't read it, do so!) It describes the search for secure foundations of mathematics through the eyes of Bertrand Russell.



The comic clearly suggests that many of those involved in the search (including Cantor, Frege, Gödel and Wittgenstein) were seriously odd.

Why does it matter?

Here are a few reasons why we might want to take a long hard look at infinity.

- Our technological society is underpinned by mathematics to an extraordinary degree now. From predicting the effects of our industrial output on future climate change, to improving the behaviour of our computers and the fuel efficiency of our transport, mathematics is needed. Now mathematics itself is unthinkable without the idea of infinity. Our number systems are infinite, the processes of calculus deal in infinitesimals; even the consistency of the body of established mathematics could be at stake.
- Until recently, every improvement in astronomical science or technology had expanded our universe. But now perhaps we seem to be coming up against some limits. Is the universe finite or infinite? How could anyone not want an answer to Māluṅkyaputta's question?