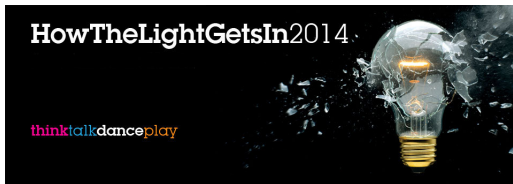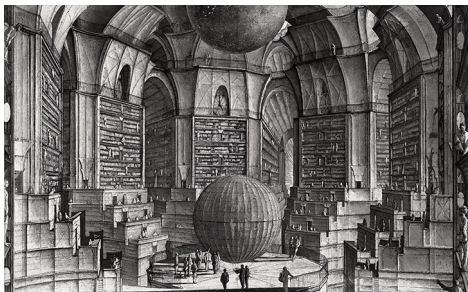# Infinity: is the universe infinite?

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The Library of Babel

The influential Argentine writer Jorge Luis Borges was fascinated by questions about mathematics, especially about infinity, which provide the basis for many of his stories.
I want to start with the story "The Library of Babel". The library is described as follows.

The universe (which others call the Library) is composed of an indefinite and perhaps infinite number of hexagonal galleries, with vast air shafts between, surrounded by very low railings. From any of the hexagons one can see, interminably, the upper and lower floors. The distribution of the galleries is invariable. Twenty shelves, five long shelves per side, cover all the sides except two; their height, which is the distance from floor to ceiling, scarcely exceeds that of a normal bookcase.

There are five shelves for each of the hexagon's walls; each shelf contains thirty-five books of uniform format; each book is of four hundred and ten pages; each page, of forty lines, each line, of some eighty letters which are black in color. . . . The orthographical symbols are twenty-five in number.

Information

As Borges points out in his story, if the library contains exactly one copy of every possible book (under the specifications outlined above), then it is after all not infinite, just unimaginably large.
I have to say that this distinction, though clear to mathematicians, is not always so to cosmologists.
If the basic currency of the universe is, as some have suggested, not space and time, but information, then perhaps the universe, like Borges' library, is finite.

Infinite in both directions?

The question whether the Universe is infinite divides into three completely different subquestions:

- ▶ Is the universe infinitely large? Does it extend infinitely in space?
- ▶ Is the universe eternal? Does it extend infinitely in time, and if so, in which direction (past or future)? or both?
- ▶ Can space (and/or time) be infinitely subdivided? That is, does the universe extend down to infinitely small scales?

Science has something to say about all three questions, but we haven't yet reached the final answers.

Linear or cyclic?

Some thinkers, most famously Nietzsche, have attempted to avoid the mental terror of an eternal universe by proclaiming the doctrine of eternal recurrence: the universe eventually returns to a previous state, and then the same things play out again.

This would only be inevitable if the universe were finite but time were unbounded. Otherwise, it is an act of faith, rather than science.

This was also treated by Borges in a lovely essay with the paradoxical title "A new refutation of time".

Eternal or not?

The majority of cosmologists believe now that the universe was created in a single event (the Big Bang) which took place roughly 1.4×10^{10} years ago.

The evidence for this is the observation that the universe is filled with almost-uniform background microwave radiation, corresponding to an observed temperature of a few degrees above absolute zero. This would result from the much hotter radiation of the Big Bang, with its wavelength stretched out by the expansion of the universe.

This expansion, based on the observed red-shift of galaxies (suggesting that they are receding from us at speeds proportional to their distance), forms the other part of the evidence. Extrapolating into the past, the galaxies would appear to have been in the same place about 1.4×10^{10} years ago.

The jury is still out over whether the universe will continue for ever or will come to an end at some distant future time. Fortunately for science, there have always been mavericks (such as Fred Hoyle, and more recently Roger Penrose) who propose alternative theories. We must ensure that any scientific theory is subject to continual scrutiny.

In any case, if the age of the universe is finite, and if (as Einstein said) no physical influence can travel faster than the speed of light, then there is only a finite region from which any signal can have reached us since the Big Bang. So whether or not the Universe is infinite, the part that can have any effect on us is finite.

Of course, wormholes in the fabric of space-time might change the picture . . .

Time does not exist?

Time is very different from space. Since it is one-dimensional, any time point divides the line into past, present and future. (Deletion of a space point leaves the rest of space connected). We only ever experience the present; we have no direct evidence of past or future. However, as a species we are prone to "look before and after", as Shelley said.

Neil Young's song "After the gold rush" captures the human predicament in time very well.

Some scientists (notably Julian Barbour) have developed a physics and cosmology in which time is not basic, but an "epiphenomenon".
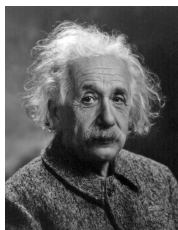
The largest scales of the universe are described by general relativity, and the smallest scaled by quantum theory. These two pillars of 20th century physics have not yet been reconciled with each other.

Infinitesimals

We already tripped over the question of the infinitely small when we considered Zeno's paradox in the last lecture. According to the paradox, Achilles comes closer and closer to the tortoise but never actually catches it.

Isaac Newton boldly used infinitesimals in his development of calculus, and his application of calculus to investigate the motions of heavenly bodies. He was taken to task by Bishop Berkeley for this. Berkeley famously compared infinitesimals to "the ghosts of departed quantities".

We'll look at his argument more closely.

Berkeley versus Newton

Berkeley considered the curve $y = x^3$. Take two neighbouring points on the curve, say (x, x^3) and (z, z^3). The slope of the line joining these two points is $(z^3 - x^3)/(z - x) = z^2 + xz + x^2$. To find the slope of the curve at the point (x, x^3), simply put $z = x$ in the last expression and find $3x^2$.

But, Berkeley remarks, you can't do that, because the division is only possible if z is not equal to x (otherwise it is $0/0$). Unless $z - x$ is one of these mysterious "ghosts of departed quantities" which remain when $z - x$ becomes zero.
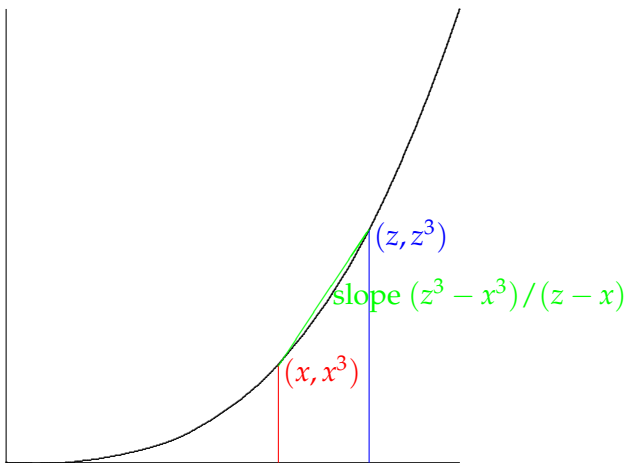
Newton claims that the slope of the curve at (x, x^3) is the slope of the green line when the blue point is moved to coincide with the red point.

Calculus

Mathematicians in the nineteenth century, notably Cauchy and Weierstrass, have given satisfactory answers to Berkeley's objections, by developing the notion of limit, and the modern treatment of calculus.

In terms of the picture just drawn, there is a line through the red point to which the green secant can be made arbitrarily close by moving the blue point sufficiently close to the red point.

The points never become "infinitesimally close together"; everything is phrased in terms of small positive differences.

Infinitesimals in physics

However, physicists still have a problem.
If you distribute a fixed electric charge over the surface of a sphere, the sphere will carry some energy caused by the electrostatic repulsion. The smaller the sphere, the larger the energy. If we allowed an electron, say, to become arbitrarily small, then sooner or later it will have more energy than the total amount in the universe!
So it seems like the electron cannot be regarded as a point charge . . .
The traditional way to deal with this is simply to subtract the infinities that arise and assume that they disappear. In recent times, string theory claims to overcome this difficulty.

Discreteness

We seem to be forced into a position that space is not infinitely divisible, that infinitesimals do not exist in nature (no matter what Cauchy and Weierstrass say about mathematics). Nobody is quite sure how this should be done, but perhaps there is a smallest possible scale of space; a piece of space of this size cannot be further cut up. (Physicists even have a candidate for this smallest possible scale, which they call the Planck length: it is about $1.61619926 \times 10^{-35}$ metres.)
Two of the contenders for a theory of quantum gravity, loop quantum gravity and causal set theory, postulate explicitly the discreteness of space. Others such as string theory have discreteness as a consequence.

Other sources of discreteness

There are other ways in which discreteness arises in continuous systems.

One was discovered in the earliest days of quantum theory. The spectrum of an operator is a collection of numbers which describe the intrinsic behaviour of the operator. The spectrum may be discrete even if the operator is continuous. The use of the word reminds us that this explains the discrete spectra of chemical elements.

Another is the notion of Hopf bifurcations in dynamical systems, which I can't discuss here.

Discreteness and information

There is a quantity which is certainly discrete: information (in the sense of computer science), which is measured in bits, or binary digits 0 and 1.

This is so commonplace we don't even notice us now. A piece of music on a CD or downloaded from the internet is nothing but a long string of 0s and 1s.

The holographic principle asserts that everything we can know about a region of space-time must pass through the boundary of that region. If the boundary is discrete, then this means that the amount of information is proportional to the area of the region, rather than to its volume.

This would explain uncertainty in physics: there is a limit to what we can know about the universe.

Conclusion

If, as I have suggested, the universe is bounded (on a large scale) and is discrete (on a small scale), then it is finite in a very strong sense: there are a finite number of distinct "things" in the universe.

I like this idea. But I am under no illusion that this is the final answer. In the past, various theories of cosmology, or of psychology, have been based on imagery derived from the current scientific paradigm. Our present paradigm is information theory, so we think of the universe in these terms. But this might change . . .