

Paradox lost: Hilbert, Gödel and modern mathematics

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Frege's bad luck

A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. In this position I was put by a letter from Mr Bertrand Russell as the work was nearly through the press.

Gottlob Frege

What was this devastating communication?

Paradox

Frege's work had been undermined by **Russell's Paradox**. I will say a few words about this.

Since Frege's time, the basic objects of mathematical thought are usually taken to be sets. Now a set just consists of some objects gathered together into a collection – but this is not a definition, since we have to say what a collection is, and it becomes circular.

Russell's paradox arises from this uncritical use of the notion of a set.

Some paradoxes

THIS STATEMENT
IS FALSE

We cannot assign a truth value to this statement, since if it is true then it is false, and *vice versa*.

The Cretan seer Epimenides said “All Cretans are liars”, as reported in the New Testament (Titus 1, 12):

One of themselves [the Cretans], even a prophet of their own, said, the Cretans are always liars, evil beasts, slow bellies.

THIS STATEMENT
IS TRUE

In some ways this statement is even more worrying. Since it is not obviously contradictory, it seems innocent; but we can't tell whether it is true or false, since if it is true then it is true, and if it is false then it is false.

A couple more

Here are a couple of my favourite paradoxes.

An adjective is **autological** if it describes itself (such as 'short' or 'pentasyllabic'), and **heterological** if it doesn't (such as 'long' or 'trisyllabic').

Most adjectives are heterological: consider 'hard', 'soft', 'red', 'blue'. [But what about 'fluffy' or 'gnarled'?]

Is 'heterological' heterological?

If it is, then it isn't; and if it isn't, then it is.

The Hypergame

We consider two-player games; the rules of the game must determine whose move it is at any stage, and when the game is over. A game is **well-founded** if any play ends after a finite number of moves (draws are permitted).

The **Hypergame** is played as follows. The first player chooses a well-founded game, and the second player then moves first in that game. For example, the first player chooses chess, and the second player plays white.

Is the Hypergame well-founded? Obviously yes, since a play of the Hypergame lasts one move longer than the play of the game chosen at the first stage.

But if it is, then the first player can choose to play the Hypergame; then the second player goes first in the Hypergame, and can also choose the Hypergame; and so on. So it is not well-founded after all!

Russell's paradox

The paradox with which Russell knocked down Frege's building will look familiar by now. Recall that Frege was using the idea of a **set** as a basic notion.

Consider the set of all sets which are not members of themselves. Is it a member of itself?

If yes, then no; if no, then yes.

Although this paradox was taken very seriously, it is the result of not thinking hard enough about exactly what a set is. I don't have time here to describe how mathematicians overcame this problem, but overcome it they did!

Hilbert's programme

We must not believe those, who today, with philosophical bearing and deliberative tone, prophesy the fall of culture and accept the ignorabimus. For us there is no ignorabimus, and in my opinion none whatever in natural science. In opposition to the foolish ignorabimus our slogan shall be:

We must know — we will know!

David Hilbert

Hilbert was firmly convinced that every mathematical problem had a solution, and that with sufficient effort, we could find the solution.

Gödel's Theorem

Gödel came up with the following variant of the paradox:

THIS STATEMENT
IS UNPROVABLE

Of course he had to be very careful about what was meant by “provable”. His proof involved a very ingenious procedure for translating logical formulae into arithmetical propositions, where they may or may not be provable.

Gödel's proof

A little background about the language here. “Provable” and “unprovable” refer to a formal system for carrying out proofs in a mechanical way, as Hilbert wanted. “True” and “false” refer to their interpretation in a given mathematical structure (in this case, arithmetic on the natural numbers).

If the axioms for arithmetic are consistent (that is, are not self-contradictory), then Gödel's statement must be true; for, if it were false, it would be provable, but a false statement cannot be proved from consistent axioms.

Hence, since the statement asserts its unprovability, it must be unprovable.

Using paradox

... there seems to be a paradox ... this was very thought-provoking and it led to exceptionally good theorems later on.

John Crossley

It is often stated that Gödel's Theorem killed Hilbert's programme, and showed that there cannot be certainty in mathematics. I believe this view is quite mistaken.

Gödel's second theorem

Hilbert was insistent on the idea that there should be a mathematical proof of consistency of any set of mathematical axioms. But, from his results, Gödel was able to prove that the axioms of Russell and Whitehead (or those of Peano) for the natural numbers, or indeed any axiom system at least as strong as these, cannot prove its own consistency.

Does this destroy the second part of Hilbert's program?
But why should a system be required to prove its own consistency?

Consistency

There are two ways to prove consistency of a mathematical theory:

- ▶ a formal proof of the type Hilbert envisaged;
- ▶ construction of a **model** for the theory, a mathematical structure satisfying the axioms of the theory. (If some object satisfies the axioms, they cannot be inconsistent!)

So, for example, someone who doesn't believe in the consistency of Peano's axioms cannot believe in the existence of the natural numbers.

In fact, an earlier theorem of Gödel (his **completeness theorem** for first-order logic) says that, if we can do one of these things, then we can do the other as well: he showed that it is possible to construct a model of any consistent theory.

Consequences

Rather than destroying mathematics, Gödel's theorems have some very interesting and creative consequences.

- ▶ First, since there are statements which are unprovable from Peano's axioms, we can add either such a statement or its negation to the axioms, producing two different models for them. This is very similar to what happened in geometry in the nineteenth century, with the discovery of **non-Euclidean geometry**. Now we have **non-standard models** of arithmetic. Paradoxically, these are very useful in proving results about the standard model!
- ▶ Second, we may be able to prove consistency in a different system.

Back to set theory

Set theory was eventually rescued from Russell's paradox by looking carefully at the way that the set-theoretic universe is built up, and capturing this building process in a collection of axioms. This was done by Zermelo, with a modification by Fraenkel. The resulting system is referred to as **ZF**, and is the most widely accepted base for mathematics.

Now it is possible to construct the natural numbers (and all the other objects of mathematical concern) within the system ZF. The resulting system of natural numbers is a model of Peano's axioms. We conclude:

The consistency of Peano's axioms is provable in ZF.

Consistency of ZF?

Since most of our mathematics is based on ZF, it might be good to have a proof of its consistency. Again, this cannot be done in ZF itself, but it can be done in a stronger system, obtained by adding a so-called **large cardinal axiom**.

It has to be admitted that such an axiom is not as widely accepted as ZF itself.

Gödel and modern mathematics

At the time of Gödel's theorem, the result caused a degree of pessimism. Maybe the reason we can't solve some hard problems is that they are unsolvable?

This has not turned out to be the case. Only a tiny number of results that people were actually interested in proving have turned out to be unprovable on the basis of, say, the ZF axioms. Mathematicians have gone on to prove Fermat's Last Theorem and the Poincaré conjecture, and to classify the finite simple groups, and are closing in on Goldbach's conjecture and the Twin Primes conjecture.

There is no crisis; we can confidently repeat Hilbert's warcry:

We must know — we will know!