

Regular polytopes

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This is joint work with Maria Elisa Fernandes (Aviero), Dimitri Leemans (Auckland) and Mark Mixer (Boston).

This year I spent nearly two months in Auckland, thanks to support from the Hood Fellowship, where I learned about regular polytopes and carried out the research reported here.

Polytopes are objects which have combinatorial, geometric and algebraic aspects.

I will be particularly concerned with **regular polytopes**, which are generalisations of the classical regular polyhedra in 3-space. They are polytopes which have the maximal amount of symmetry (in a precise sense), and not surprisingly their study has very close connections with group theory.

However, there are many questions here which haven't been very much considered by group theorists.

I begin with something that seems at first glance to have nothing at all to do with polytopes, but there is a connection ...

Independent generating sets

Let G be a finite group. A set $\{g_1, \dots, g_r\}$ of elements of G is **independent** if none of the elements lies in the subgroup generated by the others. It is an **independent generating set** if, in addition, the whole set generates the group G .

Thus independent generating sets resemble bases for vector spaces in elementary linear algebra. However, they do not have the nice properties of bases such as the **exchange property**, and so they are not the bases of a matroid.

In the symmetric group

Theorem (Julius Whiston, 2000)

The largest size of an independent set in the symmetric group S_n is $n - 1$; equality holds if and only if the set is an independent generating set.

In 2002, Philippe Cara and I found all the independent generating sets of size $n - 1$ in S_n , for $n \geq 7$. There are two types:

- ▶ The first type consists of the transpositions corresponding to the edges of a tree on n vertices.
- ▶ The second type contains one transposition; the other elements are 3-cycles and double transpositions. These will not be relevant in what follows.

There are a few extra types for small n . For example, for $n = 6$, we can take images of the above types under the outer automorphism of S_6 .

Subgroup lattices

Let $L(G)$ denote the subgroup lattice of the group G .

Proposition

For any finite group G , the Boolean lattice $B(r)$ is embeddable as a meet-semilattice of $L(G)$ if and only if it is embeddable as a join-semilattice of $L(G)$. The largest number r for which these equivalent properties hold is equal to the size of the largest independent subset of G .

If $\{g_1, \dots, g_r\}$ is an independent set in G , then the subgroups generated by its subsets form a join-semilattice of $L(G)$ isomorphic to $B(r)$.

Note that the above conditions are **not** equivalent to the embeddability of $B(r)$ in $L(G)$ as a **lattice**!

Polytopes

A polytope of dimension r is a generalisation of polygon (in 2 dimensions) or polyhedron (in 3 dimensions) to arbitrary dimension.

It can be regarded as a partially ordered set (the elements are the faces of various dimensions) in which all maximal chains contain $r + 2$ elements (including a bottom element \emptyset of dimension -1 and a top element of dimension r which represents the whole polytope). Each element can be assigned a unique dimension, corresponding to the position it occupies in a maximal flag. Elements of dimension 0, 1, 2 are vertices, edges, and faces.

The maximal chains are called **flags**.

We require several further conditions (see next slide).

- ▶ For $i < j < k$, if x, y, z are elements of dimensions i, j, k with $x \leq y$ and $y \leq z$, then $x \leq z$.
- ▶ If x and y have dimensions i and $i + 2$ and $x < y$, then there are just two elements z satisfying $x < z < y$.
- ▶ A strong connectedness condition: if F and G are two flags, then there is a sequence of flags beginning at F and ending at G , such that consecutive members intersect in all but one of their elements, and that $F \cap G$ is contained in every flag in the sequence.

The poset obtained by reversing the order is also a polytope, called the **dual** of the original.

If x and y are elements of a polytope with $x < y$, then the interval $[x, y] = \{z : x \leq z \leq y\}$ is itself a polytope, of dimension $\dim(y) - \dim(x) - 2$. In particular, if $\dim(y) - \dim(x) = 3$, then $[x, y]$ is a polygon.

Regular polytopes

If two flags $(x_{-1}, x_0, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_r)$ and $(x_{-1}, x_0, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_r)$ differ only in the element of dimension i , then any automorphism which fixes the first flag also fixes the second.

Hence, using the strong connectedness property, any automorphism which fixes a flag must fix every flag, and hence is the identity.

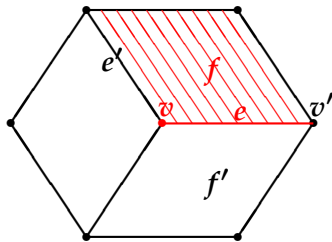
A polytope is **regular** if the automorphism group acts transitively on the flags. In this situation, the action of the group is regular: there is a bijection between flags and automorphisms. (We fix a reference flag F , and then identify F' with the unique automorphism mapping F to F' .)

If a polytope is regular, then for any i , if $\dim(x) = i - 1$, $\dim(y) = i + 2$, and $x < y$, then $[x, y]$ is a p_i -gon, where p_i depends on i but not on x and y . The vector $(p_0, p_1, \dots, p_{r-1})$ is the **Schläfli symbol** of the polytope.

String C-groups

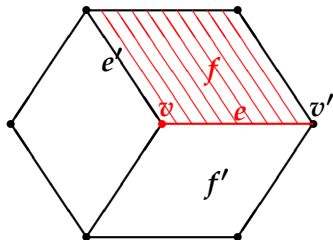
Because of the correspondence between the set of flags and the automorphism group G of a polytope, it is possible to translate everything into the group. We will see that the existence of a regular polytope is equivalent to a sequence of group elements with certain properties.

To motivate this, consider the cube.



Our reference flag is (\emptyset, v, e, f, C) (where C denotes the cube).

Let s_v, s_e and s_f be the automorphisms mapping it to (\emptyset, v', e, f, C) , (\emptyset, v, e', f, C) and (\emptyset, v, e, f', C) respectively.



Now s_v maps v' back to v , and so $s_v^2 = 1$; similarly $s_e^2 = s_{f'}^2 = 1$. Also $s_v s_e$ rotates the square face f one step clockwise, and so $(s_v s_e)^4 = 1$. Similarly $(s_e s_{f'})^3 = 1$. And s_v and $s_{f'}$ both fix e , and so they commute: $(s_v s_{f'})^2 = 1$.

More generally, we define a **string C-group** to be a finite group generated by elements s_0, s_1, \dots, s_{r-1} satisfying the conditions

- ▶ $s_i^2 = 1$.
- ▶ s_i and s_j commute if $|i - j| > 1$ (the **string condition**).
- ▶ For $I \subseteq \{0, \dots, r - 1\}$, let S_I denote the subgroup generated by $\{s_i : i \in I\}$. Then $S_I \cap S_J = S_{I \cap J}$ for any I and J (the **intersection condition**).

Theorem

The existence of a regular polytope with automorphism group G is “equivalent” (in a suitable sense) to a representation of G as a string C-group.

Note that the order of $s_i s_{i+1}$ is the i th component of the Schläfli symbol of the polytope.

We do **not** insist that s_i and s_j fail to commute if $|i - j| > 1$. In other words, we allow *degenerate* polytopes where some of the polygons are digons. This might seem to make things harder, but actually makes them much easier. The subgroup generated by a subset of $\{s_0, \dots, s_{r-1}\}$ is a string C-group in its own right, so we have the possibility of induction!

Also, we do not assume that the orders of the s_i and $s_i s_j$ give a *presentation* of a group. (If they do, then the group is a *Coxeter group*.)

Finally, the intersection condition shows that $\{s_0, \dots, s_{r-1}\}$ is an independent generating set for G . Indeed, it is stronger: it is equivalent to the condition that the map $I \mapsto G_I$ embeds the Boolean lattice $B(r)$ as a sublattice of the subgroup lattice $L(G)$ of G .

The symmetric group, 1

It follows from Whiston's theorem that the dimension of a polytope with automorphism group S_n is at most $n - 1$. It further follows from the theorem of Cameron and Cara that there is a unique such polytope of rank $n - 1$. (The condition that generators are involutions rules out the second type; the string condition shows that the tree is a string.) The generators are $s_i = (i + 1, i + 2)$ for $i = 0, \dots, n - 2$.

The corresponding polytope is the **regular $(n - 1)$ -simplex**, whose faces are all the subsets of $\{1, \dots, n\}$.

The symmetric group, 2

Fernandes, Leemans and Mixer asked about regular polytopes of smaller dimension r with group S_n . They computed the following table:

$n \setminus r$	3	4	5	6	7	8	9	10	11	12	13
5	4	1	0	0	0	0	0	0	0	0	0
6	2	4	1	0	0	0	0	0	0	0	0
7	35	7	1	1	0	0	0	0	0	0	0
8	68	36	11	1	1	0	0	0	0	0	0
9	129	37	7	7	1	1	0	0	0	0	0
10	413	203	52	13	7	1	1	0	0	0	0
11	1221	189	43	25	9	7	1	1	0	0	0
12	3346	940	183	75	40	9	7	1	1	0	0
13	7163	863	171	123	41	35	9	7	1	1	0
14	23126	3945	978	303	163	54	35	9	7	1	1

We see the entries 1 for $r = n - 1$ corresponding to the regular simplices, and we have seen that there are no more. Note also the entries 1 for $r = n - 2, n \geq 7$; 7 for $r = n - 3, n \geq 9$; 9 for $r = n - 4, n \geq 11$; and 35 for $r = n - 5, n \geq 13$.

This suggests the conjecture:

Conjecture

Given k , there is a number $N(k)$ such that, for $n \geq 2k + 3$, the number of regular polytopes of dimension $n - k$ with automorphism group S_n is $N(k)$.

Fernandes, Leemans and Mixer have established this conjecture for $k \leq 4$, with the values of $N(k)$ given above.

The alternating groups

We saw that regular polytopes with a given group (like S_n) can be studied by induction, using the fact that any subset of the generators of the string C-group themselves generate a smaller string C-group.

Fernandes, Leemans and Mixer examined the alternating group A_n . They conjectured that the largest dimension of a regular polytope with group A_n is $n/2 + 1$, with equality only for $n \equiv 2 \pmod{4}$. They managed to construct examples meeting the conjectured bound.

This, incidentally, shows that there is a big difference between largest dimension of a polytope with group G , and largest independent generating set for G (which is $n - 2$ for $G = A_n$).

Other subgroups of S_n

My contribution to this problem, after working intermittently with Dimitri Leemans on this, was the following theorem. (“Number” refers to the list of transitive groups of this degree in Magma.)

Theorem

A regular polytope of rank r whose group G is isomorphic to a transitive subgroup of S_n other than S_n or A_n satisfies one of the following:

- ▶ $r \leq n/2$.
- ▶ $n \equiv 2 \pmod{4}$, $r = n/2 + 1$ and G is $C_2 \wr S_{n/2}$. The generators are explicitly known; the Schäfli type is $(2, 3, \dots, 3, 4)$.
- ▶ G is transitive imprimitive and is one of the examples appearing in the table below.
- ▶ G is primitive. In this case, G is obtained from the permutation representation of degree 6 of $S_5 \cong \text{PGL}_2(5)$ and the polytope is the 4-simplex of Schäfli type $[3, 3, 3]$.

Degree	Number	Structure	Order	Schäfli type
6	9	$S_3 \times S_3$	36	[2, 3, 3]
6	11	$2^3 : S_3$	48	[2, 3, 3]
6	11	$2^3 : S_3$	48	[2, 3, 4]
8	45	$2^4 : S_3 : S_3$	576	[3, 4, 4, 3]

We hope to be able to use this result to prove the conjectured bound for the dimension of a polytope admitting the alternating group. It may also be of use in tackling the mysterious conjecture for the symmetric group.