The random graph

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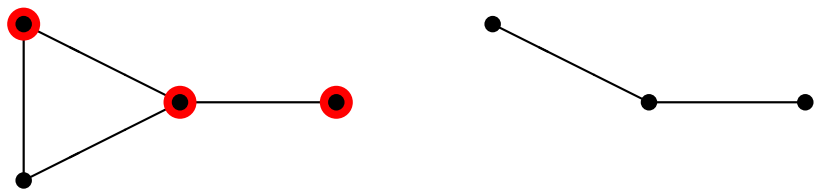
The random graph

The countable random graph is one of the most extraordinary objects in mathematics.
As well as graph theory and probability, we can turn to set theory (the Skolem paradox) or number theory (quadratic reciprocity, Dirichlet's theorem) for constructions of this object, logic ($\aleph_0$-categoricity), group theory (simple groups, Cayley graphs), Ramsey theory (Ramsey classes of structures) or topological dynamics (extreme amenability) for some of its properties, and topology (the Urysohn space) for a related structure.
I will tell you some of its story.

Graphs and induced subgraphs

A graph consists of a set of vertices and a set of edges joining pairs of vertices; no loops, multiple edges, or directed edges are allowed.



An induced subgraph of a graph consists of a subset of the vertex set together with all edges contained in the subset. In other words we are not allowed to delete edges within our chosen vertex set.

Rado's universal graph

In 1964, Richard Rado published a construction of a countable graph which was universal. This means that every finite or countable graph occurs as an induced subgraph of Rado's graph.

Rado's construction

The vertex set of Rado's graph R is the set $\mathbb{N}$ of natural numbers (including 0).
Given two vertices x and y, with $x < y$, we join x to y if, when y is written in base 2, its x-th digit is 1 – in other words, if we write y as a sum of distinct powers of 2, one of these powers is 2^x.
Don't forget that the graph is undirected! Thus

- 0 is joined to all odd numbers;
- 1 is joined to 0 and to all numbers congruent to 2 or 3 (mod 4).
- ...

Problem

Does R have any non-trivial symmetry?

The random graph

Meanwhile, Rado's fellow Hungarians Paul Erdős and Alfred Rényi showed the following theorem:

Theorem

There is a countable graph R with the following property: if a random graph X on a fixed countable vertex set is chosen by selecting edges independently at random with probability $\frac{1}{2}$, then the probability that X is isomorphic to R is equal to 1.

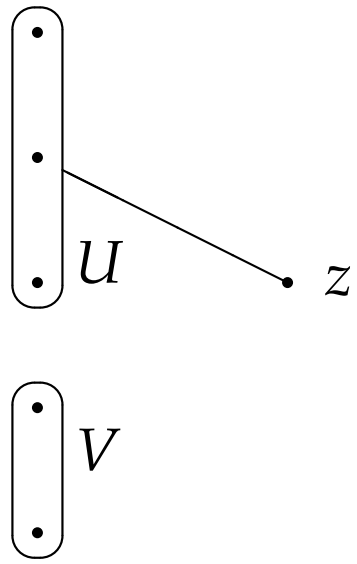
The proof

I will show you the proof.
I claim that one of the distinguishing features of mathematics is that you can be convinced of such an outrageous claim by some simple reasoning. I do not believe this could happen in any other subject.

Property $(*)$

The proof depends on the following property, which a graph may or may not possess:

$(*)$ Given two finite disjoint sets U and V of vertices, there is a vertex z which is joined to every vertex in U and to no vertex in V.



The point z is called a witness for the sets U and V.

Outline of the proof

I will prove:

Fact 1. With probability 1, a random countable graph satisfies $(*)$.

Fact 2. Any two countable graphs satisfying $(*)$ are isomorphic.

Then you will be convinced!

Proof of Fact 1

We use from measure theory the fact that a countable union of null sets is null. We are trying to show that a countable graph fails $(*)$ with probability 0; since there are only countably many choices for the (finite disjoint) sets U and V, it suffices to show that for a fixed choice of U and V the probability that no witness z exists is 0.

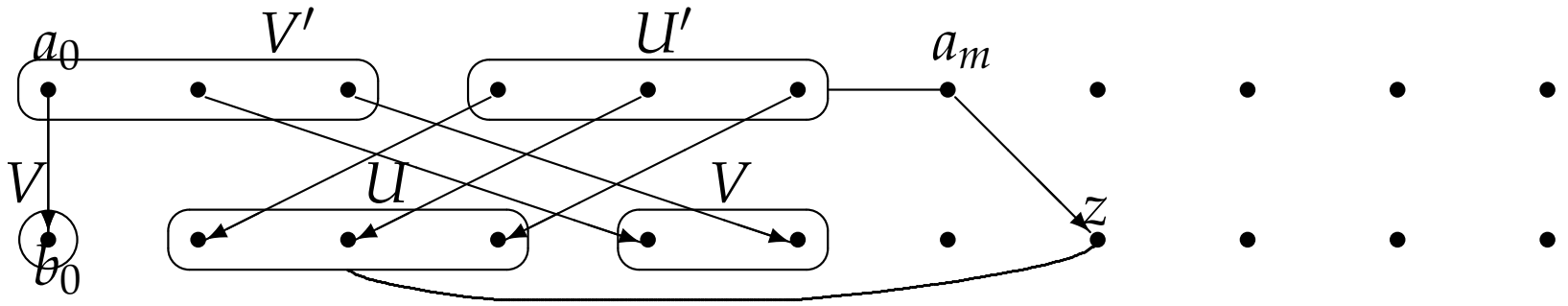
Suppose that $|U \cup V| = n$. Then the probability that a given vertex z is not the required witness is $1 - \frac{1}{2^n}$.

Since all edges are independent, the probability that none of $z_1, z_2, \ldots, z_N$ is the required witness is $\left(1 - \frac{1}{2^n}\right)^N$, which tends to 0 as $N \to \infty$.

So the event that no witness exists has probability 0, as required.

Proof of Fact 2

We use a method known to logicians as "back and forth". Suppose that Γ_1 and Γ_2 are countable graphs satisfying $(*)$: enumerate their vertex sets as $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$. We build an isomorphism ϕ between them in stages.



At stage 0, map a_0 to b_0.
At even-numbered stages, let a_m the first unmapped a_i. Let U' and V' be its neighbours and non-neighbours among the vertices alreay mapped, and let U and V be their images under ϕ. Use $(*)$ in graph Γ_2 to find a witness v for U and V. Then map a_m to z.

Fact 2, continued

At odd-numbered stages, go in the other direction, using $(*)$ in Γ_1 to choose a pre-image of the first unmapped vertex in Γ_2.
This approach guarantees that every vertex of Γ_1 occurs in the domain, and every vertex of Γ_2 in the range, of ϕ; so we have constructed an isomorphism.
The proof is finished. This is a fine example of a non-constructive existence proof: if almost all graphs have the property, then certainly a graph with the property exists. Erdős and Rényi didn't bother with an explicit construction.
Had we only gone "forward", we would only use property $(*)$ in Γ_2, and we would have constructed an embedding, but could not guarantee that it is onto.
The back-and-forth method is often credited to Georg Cantor, but it seems that he never used it, and it was invented later by E. V. Huntington.

Properties of R

Recall that a countable graph Γ is universal if every finite or countable graph can be embedded into Γ as induced subgraph.

Fact 3. R is universal (for finite and countable graphs).

To see this, revisit the back-and-forth "machine" but use it only in the forward direction. As we saw, this only requires $(*)$ to hold in Γ_2, and delivers an embedding of Γ_1 in Γ_2.

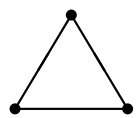
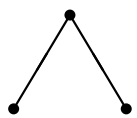
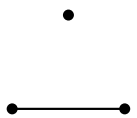
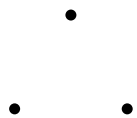
A graph Γ is homogeneous if every isomorphism between finite induced subgraphs of Γ can be extended to an automorphism of Γ. (This is a very strong symmetry condition.)

Fact 4. R is homogeneous.

To see this, take $\Gamma_1 = \Gamma_2 = R$, and start the back-and-forth machine from the given finite isomorphism.

Randomness and symmetry

The fact that the random graph is highly symmetric is surprising, for several reasons.
First, for finite graphs, the more symmetric a graph, the smaller its probability of occurrence:

Graph				
Symmetries	6	2	2	6
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

In fact, the probability of any non-trivial symmetry tends rapidly to 0 as the number of vertices increases.

Second, recall the definition of Rado's graph R:

- Vertex set $\mathbb{N}$
- For $x < y$, x and y joined if the x-th binary digit of y is 1.

I mentioned the problem of finding a non-trivial symmetry of this graph. There seems to be no simple formula for one! Rado's graph is indeed an example of the random graph. To prove this, all we have to do is to verify condition $(*)$. This is a straightforward exercise.

A number-theoretic construction

Since the prime numbers are "random", we should be able to use them to construct the random graph. Here's how.
Recall that, if p is an odd prime not dividing a, then a is a quadratic residue (mod p) if the congruence $a \equiv x^2 \pmod{p}$ has a solution, and a quadratic non-residue otherwise. A special case of the law of quadratic reciprocity, due to Gauss, asserts that if the primes p and q are congruent to 1 (mod 4), then p is a quadratic residue (mod q) if and only if q is a quadratic residue (mod p).
So we can construct a graph whose vertices are all the prime numbers congruent to 1 (mod 4), with p and q joined if and only if p is a quadratic residue (mod q): the law of quadrtic reciprocity guarantees that the edges are undirected.

This graph is isomorphic to the random graph!
To show this we have to verify condition $(*)$. So let U and V be finite disjoint sets of primes congruent to 1 (mod 4). For each $u_i \in U$ let a_i be a fixed quadratic residue (mod u_i); for each $v_j \in V$, let b_j be a fixed quadratic non-residue mod v_j.
By the Chinese Remainder Theorem, the simultaneous congruences

- $z \equiv a_i \pmod{u_i}$ for all $u_i \in U$,
- $z \equiv b_j \pmod{v_j}$ for all $v_j \in V$,
- $z \equiv 1 \pmod{4}$,

have a solution modulo $4 \prod u_i \prod v_j$. By Dirichlet's Theorem, this congruence class contains a prime, which is the required witness.

The Skolem paradox

The downward Löwenheim–Skolem theorem of model theory says that a consistent theory in a countable first-order language has a countable model.

The Skolem paradox is this: There is a theorem of set theory (for example, as axiomatised by the Zermelo–Fraenkel axioms) which asserts the existence of uncountable sets. Assuming that ZF is consistent (as we all believe!), how can this theory have a countable model?

My point here is not to resolve this paradox, but to use it constructively.

A set-theoretic construction

Let M be a countable model of the Zermelo–Fraenkel axioms for set theory. Then M consists of a collection of things called "sets", with a single binary relation $\in$, the "membership relation".
Form a graph on the set M by joining x and y if either $x \in y$ or $y \in x$.
This graph turns out to be the random graph!
Indeed, the precise form of the axioms is not so important. We need a few basic axioms (Empty Set, Pairing, Union) and, crucially, the Axiom of Foundation, and that is all. It does not matter, for example, whether or not the Axiom of Choice holds.

Back to Rado's graph

In the set-theoretic construction, it doesn't matter whether the axiom of infinity holds or not.
There is a simple description of a model of set theory in which the negation of the axiom of infinity holds (called hereditarily finite set theory). We represent sets by natural numbers. We encode a finite set $\{a_1, \ldots, a_r\}$ of natural numbers by the natural number $2^{a_1} + \cdots + 2^{a_r}$. (So, for example, 0 encodes the empty set.)
When we apply the construction of "symmetrising the membership relation" to this model, we obtain Rado's description of his graph!

Group-theoretic properties

Here are some properties of the graph R and its automorphism group.

- $\mathrm{Aut}(R)$ has cardinality $2^{\aleph_0}$.
- $\mathrm{Aut}(R)$ is simple.
- $\mathrm{Aut}(R)$ has the strong small index property: this means that any subgroup of this group with index strictly smaller than $2^{\aleph_0}$ lies between the pointwise and setwise stabilisers of a finite set.
- As a consequence, any graph Γ on fewer than $2^{\aleph_0}$ vertices satisfying $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(R)$ is isomorphic to R.
- All cycle structures of automorphisms of R are known.
- R is a Cayley graph for a wide class of countable groups, including all countable abelian groups of infinite exponent. For these groups, a "random Cayley graph" is isomorphic to R with probability 1.

Cyclic shifts

I will illustrate the above with one simple but typical argument.
A countable graph Γ admits a cyclic shift (an automorphism permuting all the vertices in a single cycle) if and only if there is a set S of positive integers such that the vertex set of Γ can be taken to be $\mathbb{Z}$, and u is joined to v if and only if $|v-u| \in S$. (The cyclic shift is simply $v \mapsto v+1$.
Now it can be shown that graphs Γ and Γ', with cyclic shifts σ and σ', give rise to the same set S if and only if

- Γ is isomorphic to Γ';
- σ and σ' are conjugate in $\mathrm{Aut}(\Gamma)$.

Now choose a set S of positive integers at random. With probability 1, the resulting graph satisfies $(*)$, and so is isomorphic to R.
Thus, R has $2^{\aleph_0}$ non-conjugate cyclic shifts!

Rolling back the years, 1

In fact, fifteen years earlier, Roland Fraïssé had asked the question: which homogeneous relational structures exist? Fraïssé defined the age of a relational structure M to be the class $\mathrm{Age}(M)$ of all finite structures embeddable in M (as induced substructure). In terms of this notion, he gave a necessary and sufficient condition.

Fraïssé classes and Fraïssé limits

Theorem

A class $\mathcal{C}$ of finite structures is the age of a countable homogeneous relational structure M if and only if

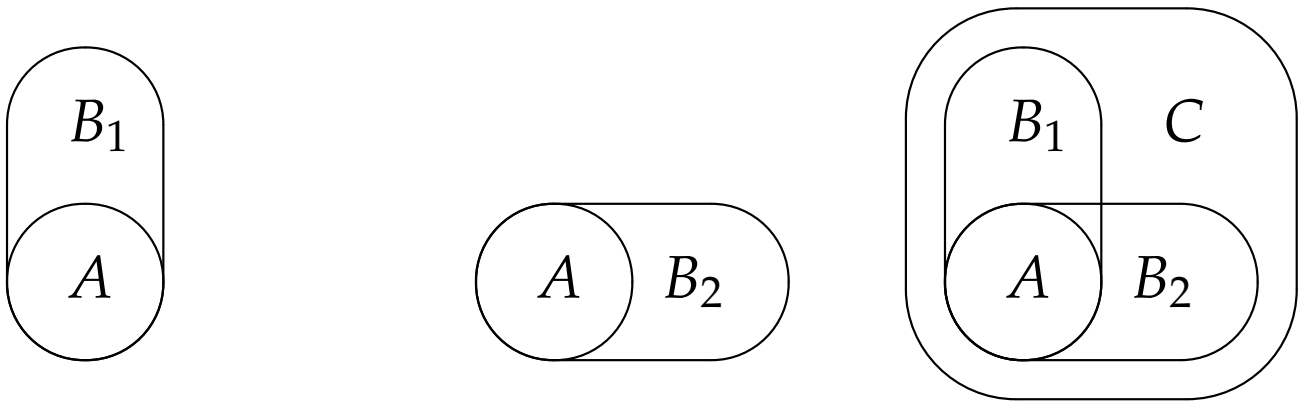
- *$\mathcal{C}$ is closed under isomorphism;*
- *$\mathcal{C}$ is closed under taking induced substructures;*
- *$\mathcal{C}$ contains only countably many non-isomorphic structures;*
- *$\mathcal{C}$ has the amalgamation property (see next slide).*

If these conditions hold, then M is unique up to isomorphism.

A class $\mathcal{C}$ satisfying these conditions is a Fraïssé class, and the countable homogeneous structure M is its Fraïssé limit.

The amalgamation property

The amalgamation property says that two structures B_1, B_2 in the class $\mathcal{C}$ which have substructures isomorphic to A can be "glued together" along A inside a structure $C \in \mathcal{C}$:



Note that the intersection of B_1 and B_2 may be larger than A.

Examples

Each of the following classes is a Fraïssé class; the proofs are exercises. Thus the corresponding universal homogeneous Fraïssé limits exist.

Fraïssé class	Fraïssé limit
Graphs	Rado's graph
Triangle-free graphs	Henson's graph
Graphs with bipartition	Generic bipartite graph
Total orders	$(\mathbb{Q}, <)$
Partial orders	Generic poset
Permutation patterns	Generic biorder

There are *many* others!

Rolling back further

A quarter of a century earlier, these ideas had already been used by the Soviet mathematician P. S. Urysohn. He visited western Europe with Aleksandrov, talked to Hilbert, Hausdorff and Brouwer, and was drowned while swimming in the sea at Batz-sur-Mer in south-west France at the age of 26 in 1924. Among his many contributions to topology was the theorem discussed below. The paper was completed from Urysohn's unpublished work by Aleksandrov and published in 1926.

Urysohn's theorem

A Polish space is a metric space which is complete (Cauchy sequences converge) and separable (there is a countable dense set). A metric space M is homogeneous if any isometry between finite subspaces extends to an isometry of M.

Theorem

There exists a homogeneous Polish space containing a copy of every finite metric space, and it is unique up to isometry.

This unique metric space is known as the Urysohn space. Its study has been popularised in recent years by Anatoly Vershik.

The proof

Here, in modern terminology, is what Urysohn did.
We cannot apply Fraïssé's Theorem directly to obtain this result, since there are uncountably many 2-element metric spaces up to isometry (one for each positive real number).
Instead, use the class of finite rational metric spaces (those with all distances rational). This is a Fraïssé class, whose Fraïssé limit is a countable universal homogeneous rational metric space.
Its completion is easily seen to be the required Polish space.

Related constructions

Various other types of metric spaces form Fraïssé classes. These include

- The class of integral metric spaces, those with all distances integers. The Fraïssé limit is a kind of universal distance-transitive graph.
- The class of metric spaces with all distances 1 or 2. The Fraïssé limit is the random graph!

Let M be the Fraïssé limit of the class of metric spaces with all distances 1 or 2; form a graph by joining two points if their distance is 1. Since the graph is homogeneous, if v and w are two vertices at distance 2, there is a vertex at distance 1 from both. Thus the distance in M coincides with the graph distance in this graph. The graph is universal and homogeneous, and so is R.

Recent developments

Recently, very close connections have emerged between countable homogeneous relational structures including a total order, Ramsey classes, and topological dynamics. Time does not permit a detailed account of this material, unfortunately.
In brief: every Ramsey class which includes a total order in its structure is the age of a homogeneous structure, and the automorphism groups of such structures are extremely amenable: that is, any continuous action on a compact space has a fixed point.
An example of such a structure is the random graph with a dense total order without endpoints of its vertex set, the graph and the order being independent.