Measuring triangle-free graphs

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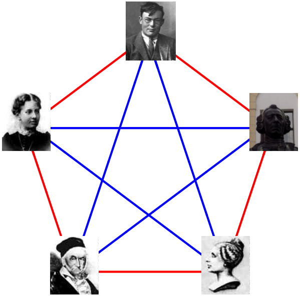
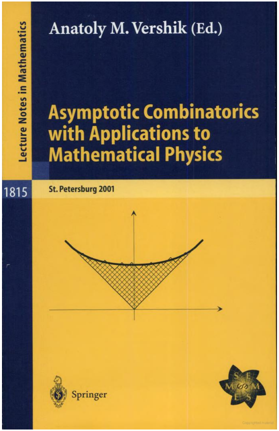
From parties . . .

Three weeks ago I was in Leuven, presenting prizes to the Flemish Mathematical Olympiad prizewinners, and giving a talk. The talk started with Ramsey theory (here is a party showing that $R(3,3) > 5$):



. . . to diagrams

. . . and ended with the shape of a random Young diagram in the Plancherel measure:



Happy birthday, Anatoly!

This talk was inspired by a conversation I had with Anatoly in Penderel's Oak, a pub in Holborn, London, about three years ago. The two of us had come to the problem of measuring triangle-free graphs from different directions: his solution (with Fedor Petrov) led to some very nice connections. But it doesn't completely answer my questions, so there is still more to be done!

The Higman–Sims graph

As a PhD student, I spent a long time thinking about the Higman–Sims graph and its automorphism group, and possible generalisations.

This graph is constructed from the Witt design, a configuration of 22 points and 77 blocks with automorphism group $M_{22}.2$.

The graph has vertex set $\{*\} \cup P \cup B$, where P and B are the point and block sets; we join $*$ to every point, a point and block whenever they are incident, and two blocks whenever they are disjoint.

The graph contains no triangles.

If we want to get a triangle-free graph by such a construction, it is necessary that we join blocks only if they are disjoint. The Higman–Sims graph is remarkable in that the converse holds.

Henson's graph

When I started thinking about the infinite in the mid-1970s, Henson's graph seemed like an obvious analogue of the Higman–Sims graph:

- it is triangle-free;
- it has the property, in terms of the previous construction, that blocks are adjacent if and only if they are disjoint;
- it is highly symmetric: indeed, it is homogeneous (this means that any isomorphism between finite subgraphs extends to an automorphism);
- it is universal: it contains every finite or countable triangle-free graph as an induced subgraph.

The random graph

Another beautiful countable object is the countable random graph, or Rado's graph.
This graph R is universal for the class of all finite (and countable) graphs (this is the property that interested Rado), and is homogeneous. Rado gave an explicit construction.
Erdős and Rényi showed a spectacular property, which gives the graph its alternative name. If we take a countable vertex set, and choose edges independently with probability $1/2$ from the set of 2-element sets of vertices, then with probability 1 the resulting random graph is isomorphic to R — a lovely example of a non-constructive existence proof!
I became interested in the automorphism group of R.

Recognising homogeneous universal structures

Fraïssé gave a test for the existence of a homogeneous relational structure M which is universal for a given class $\mathcal{C}$ of finite structures. Briefly: $\mathcal{C}$ should be the class of finite structures embeddable in M; and if $A, B \in \mathcal{C}$ with $A \subseteq B$ and $|B| = |A| + 1$, then any embedding of A into M can be lifted to an embedding of B into M.
This is sometimes called the Alice's Restaurant property, since

> *You can get anything you want*
> *At Alice's Restaurant,*

according to Arlo Guthrie: you can "order" a new point with any consistent relationships with the finitely many points you have already.

. . . and beyond

In fact, this idea goes back to P. S. Urysohn in 1924, with his construction (published posthumously) of a universal homogeneous Polish space (a complete separable metric space). It was Anatoly Vershik who told me about the Urysohn space and explained these connections to me, after my talk at the European Congress of Mathematics in Barcelona in 2000. (This was my first meeting with him.)
A Polish space is too big to apply Fraïssé's method directly. Uryshon realised that he could construct a universal homogeneous metric space with all distances rational, and then take its completion to obtain the required Polish space.
Indeed, if we replace "all distances rational" with "all distances 1 or 2", we obtain precisely the random graph!

Interlude

The symmetric group $\mathrm{Sym}(\mathbb{N})$ acts on the measure space of all graphs with vertex set $\mathbb{N}$ by measure-preserving transformations.

The action is almost highly transitive: in the induced action on the set of n-tuples of graphs, there is a single orbit with full measure. (This is because of the random 2^n-edge-colouring R_{2^n} of the complete graph.)

In particular, if I understand correctly, the action is totally non-free in the sense of Vershik: almost all pairs of graphs have stabiliser R_4, a proper subgroup of the individual stabilisers. (Actually the group is uncountable, so the theory doesn't really apply.)

I have no idea what to do with these observations …

Cyclic automorphisms

Does R have cyclic automorphisms?
We can answer this in Erdös–Rényi style as follows. A graph with a cyclic automorphism can be described by giving a set S of positive integers: take the vertex set to be $\mathbb{Z}$, and join x and y if $|x - y| \in S$.
Conversely we can extract the set S from the graph with its cyclic automorphism: number the vertices with integers so that the automorphism is the shift, and then let S be the set of positive neighbours of 0.
Now it is easy to show that two graphs with cyclic automorphisms give rise to the same set S if and only if

- the graphs are isomorphic;
- the cyclic automorphisms are conjugate.

Cyclic automorphisms of R

Choose a random subset of $\mathbb{N}$. It is very easy to show that the resulting graph satisfies Fraïssé's condition, and so is isomorphic to R.

Hence we have shown, entirely painlessly, that R has $2^{\aleph_0}$ non-conjugate cyclic automorphisms.

The technique can be used to do more, for example

- describe the cycle structure of all automorphisms of R;
- give a condition on a countable group G for R to be a Cayley graph for G.

The proofs give more. For example, if R is a Cayley graph for G, then a random Cayley graph for G is isomorphic to R with probability 1.

. . . and the Uryshon space?

Anatoly and I were able to adapt the method to show that the "rational Urysohn space" has a transitive cyclic isometry.
Hence the usual Urysohn space has a cyclic isometry with all orbits dense.
Its closure in the isometry group is an transitive abelian group, giving an abelian structure to the space (indeed many different abelian group structures).
Questions remain, for example: which abelian groups can act in this way?

What about Henson's graph?

Erdős, Kleitman and Rothschild showed that almost all finite triangle-free graphs are bipartite.
So strong is this tendency that it seems to upset any attempt to use the methods of Erdős and Rényi to study Henson's graph.
One can try various definitions of randomness for triangle-free graphs, for example:

- Take the probability of occurrence of a given finite graph on a given set of vertices to be its limiting frequency in large triangle-free graphs. By EKR, the resulting graph is almost surely bipartite (and is the unique universal "almost homogeneous" bipartite graph).
- Add edges one at a time randomly, but only add an edge if it doesn't contain a triangle. The result depends on the order in which we consider pairs of vertices.

Baire category

Help is at hand!

As well as being a measure space, the class of all graphs on the vertex set $\mathbb{N}$ is a complete metric space: two graphs are close together if they coincide on a long initial segment of $\mathbb{N}$.

Now in a complete metric space, the Baire Category theorem gives us another notion of largeness: a set is residual if it contains a countable intersection of open dense sets. These topological conditions are easily translated into graph-theoretic terms.

Now the random graph (resp., Henson's graph) is residual in the set of all countable graphs (resp., all countable triangle-free graphs).

These ideas can be applied to automorphisms. All the previous results about automorphisms of R can be proved using Baire category instead of measure.

Cyclic automorphisms of Henson's graph

Let me consider the case of cyclic automorphisms of Henson's graph in more detail.
A cyclic graph defined by a set S of positive integers is triangle-free if and only if S is sum-free: that is, if $x, y \in S$, then $x + y \notin S$.
Moreover, the graph is isomorphic to Henson's graph if and only if S is sf-universal: this means that a given finite binary word w occurs in the characteristic function of S if (and only if) w does not contain 1s in positions whose distance belongs to S.
Now sf-universal sets are residual in the collection of sum-free sets. So Henson's graph has $2^{\aleph_0}$ non-conjugate cyclic automorphisms!

Random sum-free sets

To obtain this result using measure, we would need to attach a meaning to "a random sum-free set" so that almost surely (or at least with positive probability) the random sum-free set is sf-universal.

I don't know how to that, but I spent some time playing with a very simple model.

Consider the natural numbers in order. When considering n, if $n = x + y$ with $x, y \in S$, then of course $n \notin S$; otherwise, decide whether $n \in S$ by the toss of a fair coin.

It is not surprising to learn that the probability that S consists entirely of even numbers is zero. However, there is a surprise in store!

Odd numbers

Theorem

The probability that the random sum-free set S consists entirely of odd numbers is non-zero: in fact this probability is approximately 0.218.

Intuitively the reason is clear. If after many steps we have only obtained odd numbers, then the next even number is very likely to be excluded as the sum of two numbers previously chosen, but the next odd number still gets included with probability $1/2$.

In some sense this is the EKR theorem rearing its head again: if S consists of odd numbers only, then every edge in the Cayley graph of $\mathbb{Z}$ with respect to S joins an even number to an odd number, so the graph is bipartite.

This was the first theorem I proved with the help of a computer (a Sinclair Spectrum with 48 kilobytes of RAM and 3.5Mhz clock speed).

More fragments of the space

A subset of $\mathbb{Z}/(n)$ is complete sum-free if it is sum-free and every $z \notin S$ can be written as $z = x + y$ where $x, y \in S$.
So $\{1\}$ (mod 2) (the odd numbers) gives an example, but there are many others, such as $\{1, 4\}$ (mod 5) or $\{2, 3\}$ (mod 5).

Theorem

- *The probability that S is contained in a given complete sum-free set is non-zero.*
- *More generally, if T (mod n) is a complete sum-free set and A a finite set of natural numbers, then the probability that S is contained in the union of T and A is non-zero.*

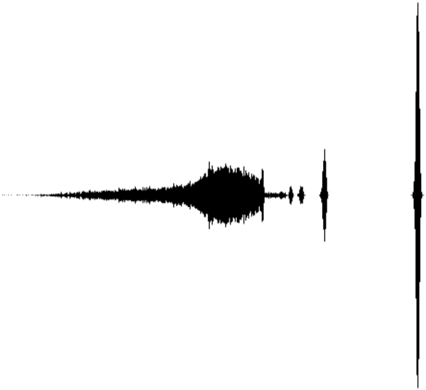
An example of the last case is the set of sum-free sets in which 2 is the only even number, which has probability somewhere round 10^{-6}.

Density

In order to visualise this complicated structure, one can consider the density of a random sum-free set. (I conjecture that the density exists with probability 1.)
If $T \pmod{n}$ is complete sum-free, then elements of T occur with probability close to $1/2$, so the density is almost surely $|T|/2n$. This gives $1/4$ for sets of odd numbers, $1/5$ for sets contained in $\{1, 4\} \pmod 5$ or $\{2, 3\} \pmod 5$, etc.
Plotting the density of large finite sum-free sets is like using a spectroscope: the longer you wait, the more accurate the plot should be. We expect a spectral line at $1/4$ with intensity $0.218\ldots$, and weaker lines at $1/5$, $3/16$, and so on.

Density plot

A plot of the density of about 10^6 subsets of $[1, 10^5]$ looks like this:



Questions

This plot raises various questions:

- First, as mentioned, does the density exist almost surely?
- Is the density positive almost surely?
- Is the spectrum discrete above $1/6$?
- What happens below $1/6$? Is there a continuous part to the spectrum, or is it many discrete parts smeared together?

One thing we do know. Tomasz Schoen proved:

Theorem

A sf-universal set has density 0.

So probably this model does not give information about Henson's graph.

Further development

The most important development was the result of Petrov and Vershik mentioned in the introduction. These authors constructed an exchangeable measure on countable triangle-free graphs, which is concentrated on the isomorphism class of Henson's graph.

Their approach was quite different. They constructed an uncountable graph on the unit interval in which triangles have measure zero, and then obtained the countable random graph by sampling vertices from the unit interval. There seems to be some connection with the Lovász–Szegedy theory of graphons.

The method has been extended to a very wide range of homogeneous relational structures by Nate Ackerman, Cameron Freer and Rihanna Patel.

Cayley graphs for other groups

As mentioned earlier, there is a near-characterisation of countable groups which admit the randm graph as a Cayley graph; there are necessary and sufficient conditions are bit complicated to state, but all countable abelian groups of infinite exponent satisfy them. Moreover, if some Cayley graph is isomorphic to R, then almost all are.

This result can also be proved by Baire category arguments.

Also, Baire category arguments can be used for Henson's triangle-free graph, showing it to be a Cayley graph for a wide variety of groups.

The methods fail for Henson's universal homogeneous K_n-free graph for $n > 3$: this graph is not a normal Cayley graph for any group, and in particular is not a Cayley graph for any abelian group.

Recently Greg Cherlin showed that these graphs are Cayley graphs for some groups including non-abelian free groups.

What I would like

For the reasons explained, I would like a measure on the class of sum-free subsets of $\mathbb{N}$ which is concentrated on the sf-universal sets.

More generally, I want a measure on the set of pairs consisting of a triangle-free graph on $\mathbb{N}$ and a group acting on $\mathbb{N}$ by graph automorphisms, which is concentrated on the isomorphism class of Henson's graph. (I talked only about the infinite cyclic group above, but that was only the first step.)

Also I would like more general results. I would like to be able to show the known results about the random graph and Henson's graphs as Cayley graphs by a measure-theoretic method.

This must somehow involve choosing at random a structure which encodes the group G and the connection set S from which the Cayley graph is built.

Cherlin's results, using bare-handed constructions, will probably be harder than the results for the triangle-free case.

Almost highly transitive?

Is the infinite symmetric group almost highly transitive on the set of graphs if we use the Petrov–Vershik triangle-free measure? In other words, if we pick n random graphs from this measure, does the n-tuple form a unique configuration?

Probably the answer is yes, and the proof not too hard.

More generally, are there other almost highly transitive group actions? And what can be said about them?

The end

Thank you for your attention.
I wish Anatoly a very happy birthday, and many more years of producing deep mathematics and spreading his knowledge widely, as he has done up to now.