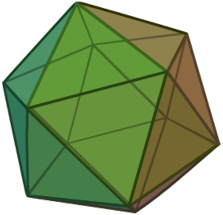
Two-graphs revisited

Peter J. Cameron

University of St Andrews

Modern Trends in Algebraic Graph Theory
Villanova, June 2014

History



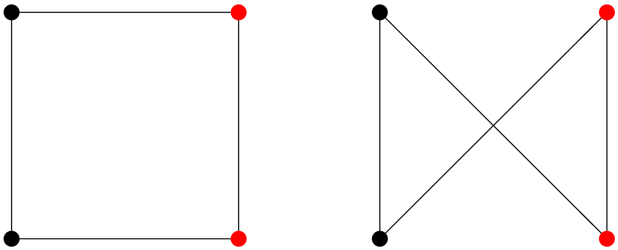
The icosahedron has six diagonals, any two making the same angle ($\arccos(1/\sqrt{5})$).
Three diagonals through the vertices of a face can be directed so that any two make an acute angle. But for the remaining triples of diagonals, this is not possible; they can be directed so that any two make an obtuse angle.
The rotation group of the icosahedron (which is isomorphic to A_5) permutes the six diagonals 2-transitively and preserves the two types of triples.

Around 1970, Graham Higman considered the Conway sporadic simple group Co_3, one of only two sporadic simple groups (apart from the Mathieu groups) to have a 2-transitive permutation action (in this case, with degree 276).

Higman gave a combinatorial construction of a set of triples taken from a set of 276 points which has Co_3 as its automorphism group. This set has the property that any four points contain an even number of distinguished triples. Higman called such a set a two-graph.

The Higman two-graph arises from a set of 276 equiangular lines in $\mathbb{R}^{23}$, which is realised inside the Leech lattice.

A little earlier, motivated by questions in elliptic geometry, specifically the congruence order of the elliptic plane, Jaap Seidel had represented a set of n equiangular lines by a graph on n vertices, or more precisely, a switching class of graphs. (The operation of switching with respect to a set of vertices interchanges edges and non-edges between the set and its complement, while leaving edges within the set or within its complement unchanged.)



The Seidel tree in Eindhoven

Equivalent concepts

It turns out that the following concepts are all, in some sense, equivalent:

- two-graphs;
- switching classes of graphs;
- double covers of complete graphs;
- sets of equiangular lines in Euclidean space.

Seidel wrote a number of surveys of two-graphs in the 1970s and 1980s. Recently there have been some new developments. I will briefly describe the equivalences and then discuss some new things.

Two-graphs and switching classes

In a graph, we say that an odd triple is a triple of vertices which contains an odd number of edges. A pair of graphs is equivalent under switching if and only if they have the same set of odd triples.

The set of odd triples in a graph is a two-graph (that is, any 4-set contains an even number of odd triples). Every two-graph arises in this way.

A consequence is that the automorphism groups of the two-graph and of the corresponding switching class of graphs are equal. (An automorphism of a switching class is a vertex permutation which carries some, and hence every, graph in the class to another graph in the class.) This group contains the automorphism group of each graph in the switching class as a subgroup.

A representation theorem

Here is a Frucht-type theorem which is a converse to the preceding remark.

Theorem

Given any finite group G, there is a switching class $\mathcal{C}$ of graphs with the properties

- $\mathrm{Aut}(\mathcal{C}) = G$;
- *for each subgroup $H \leq G$, there is a graph $\Gamma \in \mathcal{C}$ with* $\mathrm{Aut}(\Gamma) = H$.

A binary representation

Given a set X of n points, let V_k be the vector space of functions from the set of k-subsets of X to the binary field $\mathbb{F}_2$.

For any $k < n$, there is a coboundary map δ_k from V_k to V_{k+1}, where $\delta_k(f)$ is the function whose value on a $(k+1)$-set is the sum of the values of f on its k-subsets.

Now V_0 has dimension 1, and the image of δ_0 contains the two sets $\emptyset$ and X (the all-zero and all-one vectors).

The kernel of δ_1 is equal to this image, and its image is the set of complete bipartite graphs.

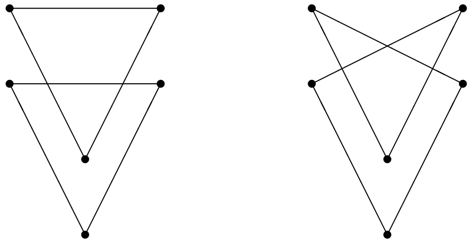
The kernel of δ_2 is equal to this image, and its image is the set of two-graphs, which is the kernel of δ_3.

These facts (easily shown directly) state that the $\mathbb{F}_2$ cohomology of the simplex vanishes in dimensions 0, 1 and 2.

Double covers of complete graphs

By a double cover of K_n I mean a graph Δ with a 2-to-1 covering map from the vertices of Δ to those of K_n, so that each edge of K_n is covered by two edges of Δ.
If we choose arbitrarily one of each pair of vertices covering each vertex of K_n, we obtain a graph Γ on n vertices. Replacing some vertices by the other vertices in their pairs has the effect of switching Γ with respect to this set of vertices.

There are two double covers of K_3, a pair of triangles and a hexagon. Taking as triples the 3-sets covered by a pair of triangles, we obtain the two-graph which corresponds to this switching class.



The automorphism group of the double cover is itself a double cover of the automorphism group of the two-graph, which may or may not split. In the case of the icosahedron, the double cover is just the 1-skeleton, and its automorphism group is $C_2 \times A_5$.

Equiangular lines

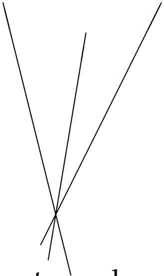
We use the Seidel adjacency matrix A of a graph, whose rows and columns are indexed by the vertices, and whose entries are 0 on the diagonal, $+1$ for adjacency, and -1 for non-adjacency. The matrix A is real symmetric, and so is diagonalisable: its eigenvalues are real.

Switching the graph corresponds to pre- and post-multiplying A by a diagonal matrix with entries -1 on the switching set and $+1$ elsewhere. So graphs in the same switching class have Seidel adjacency matrices which are similar, and so have the same Seidel spectrum.

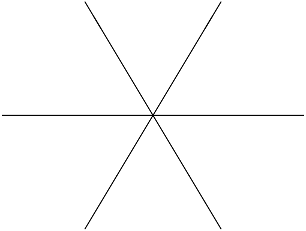
Since A has trace 0, its smallest eigenvalue is negative, say $-\lambda$, with multiplicity $n-d$, say. (Assume that the graph is not null.) Then $A+\lambda I$ is positive semi-definite, and so is the Gram matrix of inner products of a set of vectors in $\mathbb{R}^d$. Each vector has squared length λ, and the inner products of different vectors are ± 1; so the angle between any two vectors is $\arccos(1/\lambda)$ or $\pi - \arccos(1/\lambda)$. Thus the lines spanned by these vectors are equiangular.

Switching the graph corresponds to replacing the vectors in the switching set by their negatives.

Conversely, given a set of equiangular lines, choose a unit vector along each line as a vertex of a graph, where two vertices are adjacent if the vectors make an acute angle. Different choices of vectors give switching-equivalent graphs.



Acute angles



Obtuse angles

Heavily covered points

The following theorem was proved by Boros and Füredi for $d = 2$, and Bárány for arbitrary d.

Theorem

For any $d \geq 2$, there is a positive constant c_d with the property that, for any set of n points in general position in $\mathbb{R}^d$, there is a point (not necessarily one of the given points), which lies in a proportion at least c_d of the $\binom{n}{d+1}$ simplexes spanned by the $(d+1)$-sets of points in the set.

The correct value of c_2 (the supremum of all real numbers for which the statement holds) is $2/9$; but for $d > 2$, only upper and lower bounds are known.

Gromov's result

Mikhail Gromov found a procedure for improving the lower bound on the constant c_d. The method was simplified by Karasev, and an accessible exposition was given by Matoušek and Wagner.

Gromov's method for c_d involves a collection of $d-1$ functions; the first function in the list involves two-graphs, so I will just concentrate on this.

The density of a graph (or a two-graph) is the proportion of all 2-sets (resp. 3-sets) which are edges of the graph (resp. triples of the two-graph. Gromov's function ϕ_2 is defined by the rule that $\phi_2(\alpha)$ is the limit inferior of the densities of two-graphs with the property that all graphs in the corresponding switching class have density at least α.

Application

Using Razborov's flag algebra method, Kráľ, Mach and Sereni were able to find lower bounds for $\phi_2(\alpha)$ and hence for c_d for arbitrary d.
For $\alpha \in [0, 2/9]$ (the values needed for Gromov's method), they show that $\phi_2(\alpha) \geq \frac{3}{4}\alpha(3 - \sqrt{8\alpha + 1})$.
In this way, they improved the lower bound on c_3 from 0.06332 to 0.07480. (The best known upper bound is 0.09375.)

Statistics of switching classes

Computing Gromov's ϕ_2 is equivalent to finding the smallest number of edges in a graph in the switching class of a two-graph with given density. Note that the minimising graph has the property that, given any 2-partition of the vertices, at most half of the pairs crossing the partition are edges of the graph.
More general question about the distribution of numbers of edges in the switching class of a graph could be asked.
I will just make the easy remark that the average edge-density in any switching class is $1/2$, since half of the switching partitions separate any given pair of points.

Primitive groups are small

A permutation group G acting on a set X of n points (with $n > 2$) is primitive if there is no partition of X invariant under G apart from the two trivial partitions (into singletons, and with a single part).

Primitive groups have always been the main focus of attention in finite permutation group theory. Since the Classification of Finite Simple Groups (CFSG), we know much more about primitive groups. In particular, apart from symmetric and alternating groups, they have very small orders.

I will trace one measure of this, and a recent result in this line of work related to two-graphs.

A theorem of Cameron, Neumann and Saxl

In the early 1980s, with Πeter Neumann and Jan Saxl, I proved:

Theorem

If G is a primitive group on X, other than S_n and A_n and finitely many exceptions, then there is a subset of X whose setwise stabiliser in G is the identity.

Extensions

This result has been quantified in various ways:

- Ákos Seress found the finitely many exceptions: there are 43 of them, the largest degree being 32.
- I showed that the proportion of subsets whose stabiliser is trivial tends to 1 as $n \to \infty$ (in primitive groups of degree n other than S_n and A_n).
- Laci Babai and I showed that we can take the size of the subset to be at most $n^{1/2+o(1)}$.

Seress' list

Here for the record is Seress' list of primitive groups with no regular orbit on the power set, excluding symmetric and alternating groups. The first number is the degree and the second is the number of the group in the GAP library of primitive groups.
$(5, 2, D_{10})$, $(5, 3, F_{20})$, $(6, 1, A_5)$, $(6, 2, S_5)$, $(7, 4, F_{42})$, $(7, 5, L_3(2))$, $(8, 2, 2^3.7.3)$, $(8, 3, L_(2))$, $(8, 4, L_3(2).2)$, $(8, 5, 2^4.L_3(2))$, $(9, 2, 3^2.D_8)$, $(9, 5, 3^2.8.2)$, $(9, 6, 3^2.2.L_2(3))$, $(9, 7, 3^2.2.L_3(3).2)$, $(9, 8, L_2(8))$, $(9, 9, L_2(8).3)$, $(10, 2, S_5)$, $(10, 3, A_6)$, $(10, 4, S_6)$, $(10, 5, A_6.2)$, $(10, 6, A_6.2)$, $(10, 7, A_6.2^2)$, $(11, 5, L_2(11))$, $(11, 6, M_{11})$, $(12, 2, L_2(11).2)$, $(12, 3, M_{11})$, $(12, 4, M_{12})$, $(13, 7, L_3(3))$, $(14, 2, L_2(13).2)$, $(15, 4, A_8)$, $(16, 16, 2^4.(A_5 \times 3).2)$, $(16, 17, 2^4.A_6)$, $(16, 18, 2^4.S_6)$, $(16, 19, 2^4.A_7)$, $(16, 20, 2^4.L_4(2))$, $(17, 7, L_2(16).2)$, $(17, 8, L_2(16).4)$, $(21, 7, L_3(4).3.2)$, $(22, 1, M_{22})$, $(22, 2, M_{22}.2)$, $(23, 5, M_{23})$, $(24, 3, M_{24})$, $(32, 5, 2^5.L_5(2))$.

Automorphism groups of hypergraphs

Frucht showed that every group is the automorphism group of a graph. But not every permutation group is the automorphism group of a graph acting on the vertices (for example, 2-transitive groups cannot be). Using the results described earlier, Babai and I showed:

Theorem

Apart from the alternating groups and finitely many others, every primitive group is the full automorphism group (acting on vertices) of an edge-transitive hypergraph.

The problem of finding the finite list of exceptions is open. Babai and I showed that asymptotically we can take the cardinality of edges in the hypergraph to be $n^{1/2+o(1)}$.

A theorem about switching classes

I will present a new result along these lines.

It might be thought that very symmetric switching classes (say those with primitive automorphism groups) will be made up of very symmetric graphs. But in fact, we have:

Theorem

Apart from the switching classes of the complete and null graphs, and finitely many others, every switching class with primitive automorphism group contains a graph with trivial automorphism group.

Work to determine the finitely many exceptions is in progress.

Dimension

The final result is related to the concept of metric dimension. I will use a version of this more adapted to the situation.
Given a graph Γ, a graph basis of Γ is a set S of vertices with the property that distinct vertices outside S have distinct neighbour sets in S. The graph dimension of Γ is the smallest size of a graph basis.
Note that a graph basis is a base for the automorphism group of Γ (its pointwise stabiliser is trivial), so the graph dimension is an upper bound for the base size.
Similarly, given a 3-uniform hypergraph Γ, a hypergraph basis for Γ is a set S of vertices such that, for distinct vertices $v \notin S$, the graphs $\Gamma_v(S)$ with edges the pairs $\{x, y\} \subseteq S$ for which $\{v, x, y\}$ is an edge of Γ, are distinct.

Descendants

If T is a two-graph with a vertex v, there is a unique graph Γ_v in the corresponding switching class which has v as an isolated vertex. (Take any graph in the switching class, and switch with respect to the neighbours of v.) The graphs $\Gamma_v - v$ are called descendants of T.

The descendants of a two-graph are the graphs induced on vertex neighbourhoods in the corresponding double cover of the complete graph.

In particular, if two descendants are isomorphic, then the corresponding vertices are in the same orbit of the automorphism group of the two-graph; so if all descendants are isomorphic, then the automorphism group is transitive (and conversely).

Regularity

Recall that a two-graph is regular if any two vertices lie in the same number of triples. Now the following are equivalent:

- the two-graph T is regular;
- the corresponding double cover is a Taylor graph (an antipodal distance-regular graph with diameter 3 and antipodal classes of size 2);
- some (or equivalently every) descendant of T is strongly regular, with $k = 2\mu$;
- the Seidel spectrum has just two eigenvalues.

For example, the descendants of the two-graph associated with the diagonals of the icosahedron are pentagons (as can be seen by looking along a diagonal). The descendants of the Higman two-graph for Co_3 are isomorphic to the McLaughlin strongly regular graph.

A theorem on dimensions

Robert Bailey showed that the metric dimension of the descendants of any Taylor graph differ by at most 1. This suggested a general result which requires no regularity.

Theorem

- *The graph dimension of any graph in the switching class associated with a two-graph T does not exceed the hypergraph dimension of T.*
- *Let v be a vertex of a two-graph T. Then the hypergraph dimension of T is at most one more than the graph dimension of the descendant $\Gamma_v - v$.*