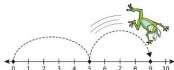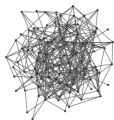# Four precious jewels

Peter J. Cameron
University of St Andrews

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Four precious jewels

My topic is four remarkable mathematical objects:

- the (Erdős–Rényi) random graph, or Rado's graph;
- the rational numbers, as ordered set;
- the Urysohn metric space;
- the pseudo-arc.

In each case, the object can be constructed and studied by methods of finite combinatorics (usually some variant on Fraïssé's amalgamation method); and they in turn contribute to areas of finite combinatorics such as Ramsey theory, as well as further afield in topological dynamics.

Finite random graphs

In 1963, Paul Erdős and Alfred Rényi wrote a paper on "Asymmetric graphs". They showed that, not only does a random finite graph (edges chosen independently with probability $\frac{1}{2}$) have no non-trivial automorphisms (with high probability), but it is in a certain sense at maximum distance from symmetry.

So beautiful symmetric objects like the Petersen graph are rare.
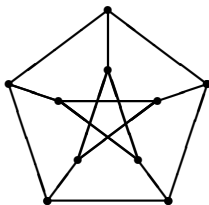
Countable random graphs

In a short section of the paper, Erdős and Rényi showed that the situation for countably infinite random graphs is different: such a graph has infinitely many automorphisms with probability 1. The reason for this is even more extraordinary: there is only one countably infinite random graph! (This is not in the paper, though the tools to prove it are developed there: it is a short appendix in the book of Erdős and Spencer on *Probabilistic Methods in Combinatorics* in 1974.)

This one remarkable graph has many more properties, and arises in many different parts of mathematics, as you might expect.

Rado's graph

In 1964, Richard Rado published a construction of a countable universal graph (one embedding every finite or countable graph as induced subgraph). Rado said nothing about uniqueness or automorphisms: indeed his construction has no obvious symmetry.

Here is Rado's construction. The vertex set of his graph is the set of natural numbers (including 0); for $x < y$, we join x to y if the xth digit in the base 2 expansion of y is equal to 1.

Exercise

Show that there is a primitive recursive automorphism interchanging the vertices 0 and 1. (You will probably not be able to discover a simple formula for such an automorphism.)

There are many other constructions of this graph. For example, take the vertices to be the primes congruent to 1 (mod 4); join p to q if p is a quadratic residue mod q.

But why is this graph isomorphic to the one given by Rado's "powers of 2" construction?

Rado's graph turns out to be "the" Erdős–Rényi random graph!

Why does it work?

The property of the graph R used both by Erdős and Rényi and by Rado is the following, sometimes called the Alice's Restaurant property ("you can get anything you want"):

($*$)   Given finite disjoint sets U and V of vertices of R, there is a vertex z joined to everything in U and to nothing in V.
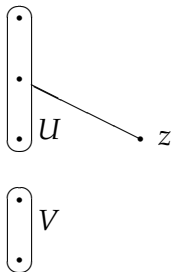

This property says that given any finite subgraph W of R, every possible extension of W to a graph with one extra vertex is realised inside R.

It is a fairly easy calculation to show that this holds with probability 1 in a countable random graph. (There are countably many choices of U and V; for each choice of U and V, z exists with probability 1; and a countable intersection of sets of measure 1 has measure 1.)

Now property $(*)$ says that any embedding of a finite graph into R can be extended to one further vertex in all possible ways. This shows that R is universal. Also, used in "back-and-forth" fashion, it shows that any two graphs with property $(*)$ are isomorphic.

The picture on the next slide demonstrates the process.
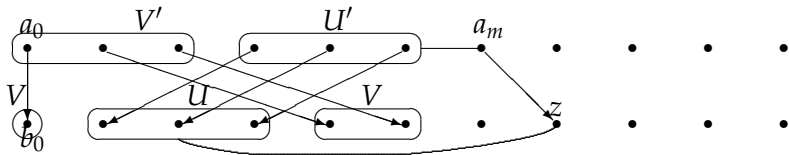
At stage 0, map a_0 to b_0. At even-numbered stages, let a_m the first unmapped a_i. Let U' and V' be its neighbours and non-neighbours among the vertices alreay mapped, and let U and V be their images under ϕ. Use $(*)$ in graph Γ_2 to find a witness v for U and V. Then map a_m to z.

To produce an isomorphism, we must go "back and forth": at odd-numbered stages, choose the first unused vertex in B, and use property $(*)$ in A to find a preimage for it.

Now, in order to show that Rado's construction and the quadratic residue construction give isomorphic graphs (and that the graph is universal), it suffices to verify the Alice's Restaurant property.

This is very easy for Rado's construction. For the other construction, it is a pleasant exercise in nineteenth century number theory (quadratic reciprocity and Dirichlet's theorem are needed).
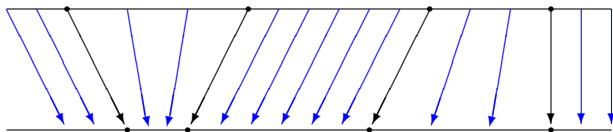
Homogeneity

A structure A is homogeneous if any isomorphism between finite substructures of A can be extended to an automorphism of A.

The random graph is homogeneous. This is shown by examining the back-and-forth proof of uniqueness. Suppose that $f : B \to C$ is an isomorphism between finite subgraphs of R. Then f can be extended to map one additional point of B into R; or its inverse can be extended to map onto one additional point of C. Then the back-and-forth method shows that f can be extended to an automorphism of the whole graph.

This is the basis for a wide-ranging study of the automorphisms and endomorphisms of R; but that is another topic!

The rational numbers

A more familiar example of a universal and homogeneous
structure is the set $(\mathbb{Q}, <)$ of rational numbers (as ordered set).
This is characterised as the unique countable dense linear order
without endpoints, by a celebrated theorem of Cantor.
Homogeneity is shown as follows. Given an isomorphism
between finite subsets, there is a piecewise-linear
order-preserving map on the whole of $\mathbb{Q}$ which carries the first
to the second:

Fraïssé

In the late 1940s and early 1950s, Roland Fraïssé gave a wide generalisation of the fact that the rational numbers are homogeneous and universal for finite and countable ordered sets. The context is relational structures over a given relational language $\mathcal{L}$, that is, sets carrying relations of the arities specified by the language. (For example, for graphs or total orders we take a single binary relation.)

Until the end of the talk, maps between structures will always be embeddings as induced substructures: that is, $f : A \to B$ carries each instance of a relation in A to an instance of the corresponding relation in B, and each instance of a relation in the image of A arises in this way. Remember: for graphs, we are using induced subgraphs.

I allow the empty set as a structure, but require it to be unique (that is, there are no "nullary relations" in the language).

The age of a relational structure A is the class of all finite structures embeddable in A as induced substructures.

A class $\mathcal{A}$ of finite structures is hereditary if it is closed under taking substructures. It has the amalgamation property if two structures B_1, B_2 in the class $\mathcal{A}$ which have substructures isomorphic to A can be "glued together" along A inside a structure $C \in \mathcal{A}$:
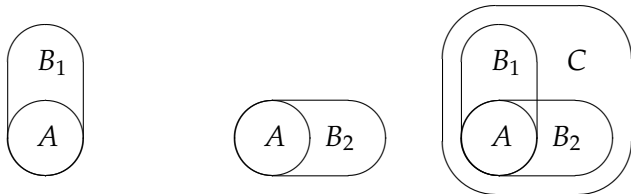
Since the empty set is a structure, I do not have to state the joint embedding property separately, it is a special case.

Fraïssé's Theorem

Theorem
A class $\mathcal{A}$ of finite structures is the age of a countable homogeneous relational structure if and only if it is closed under isomorphism, hereditary, has only countably many members up to isomorphism, and has the amalgamation property.

If these conditions hold, then the countable homogeneous structure M with age $\mathcal{A}$ is unique up to isomorphism.

A class $\mathcal{A}$ satisfying these conditions is called a Fraïssé class, and the countable homogeneous structure M is its Fraïssé limit. The Alice's restaurant property of the Fraïssé limit M says that, if B and C are elements of the age of M with $B \subseteq C$ and $|C| = |B| + 1$, then any embedding of B in M can be extended to an embedding of C. Then everything works just as for the random graph.

Examples

Each of the following classes is a Fraïssé class; the proofs are exercises. Thus the corresponding universal homogeneous Fraïssé limits exist.

Fraïssé class	Fraïssé limit
Graphs	Rado's graph
Triangle-free graphs	Henson's graph
Graphs with bipartition	Generic bipartite graph
Total orders	$(\mathbb{Q}, <)$
Partial orders	Generic poset
Permutation patterns	Generic permutation

There are *many* others!

Urysohn

Fraïssé anticipated Erdős and Rényi by more then ten years, but he was not the first to use these methods.

Pavel Samuilovich Urysohn was a Soviet pioneer of topology. He came to western Europe with Aleksandrov, and met Brouwer and Hilbert. On holiday in the south of France, he was drowned while swimming in the sea, at the age of 26. One of his last pieces of work was later written up by Aleksandrov and Brouwer.

A Polish space is a complete separable metric space: that is, one in which all Cauchy sequences converge, and there is a countable dense subset. Urysohn showed that there is a universal homogeneous Polish space: that is, every Polish space is embeddable in Urysohn's space, and any isometry between finite subsets extends to an isometry of the whole space.

We will look at this result from a post-Fraïssé viewpoint.

Urysohn space

The class of finite metric spaces is not a Fraïssé class, since there are too many: there are uncountably many 2-element metric spaces up to isometry.

Instead, Urysohn essentially showed that the finite rational metric spaces (those with all distances rational) is a Fraïssé class. Take its Fraïssé limit, and then take the completion, to get the Urysohn space.

Incidentally, if we play the same game with metric spaces with all distances integral, we obtain a homogeneous distance-transitive graph; and if we use just distances 1 and 2, we obtain the random graph!

Ramsey's theorem

Recently there has been a remarkable development connecting homogeneous structures with Ramsey classes and topological dynamics, which I now sketch.

Ramsey's theorem states:

Theorem
Given positive integers k, l, r with $k < l$, there is an integer N such that, if the k-subsets of an N-set are coloured with r different colours, there is an l-set all of whose k-subsets have the same colour.

The aim is to generalise this from a theorem about sets to a theorem about structures such as graphs. In order to do this, it is convenient to replace "subsets" by "embeddings".

Accordingly, if A and B are structures, we denote by $\binom{B}{A}$ the set of all embeddings of A into B.

Ramsey classes

We assume that all our classes of finite structures are hereditary (closed under substructures). Now we can turn Ramsey's Theorem into a definition:

A class $\mathcal{A}$ of finite structures is a Ramsey class if, given $A, B \in \mathcal{A}$ and $r > 1$, there is a structure $C \in \mathcal{A}$ such that, if elements of $\binom{C}{A}$ is coloured with r colours, there exists an element $f \in \binom{C}{B}$ (an embedding of B into C) such that all embeddings of A into C with image contained in the image of f have the same colour.

It turns out that this can only hold if all the structures in $\mathcal{A}$ have trivial automorphism group. This can be achieved by considering them as labelled structures, that is, the point set is $\{1, \ldots, n\}$ for some n (i.e. is totally ordered).

Nešetřil's programme

Jarik Nešetřil observed that, if a hereditary class $\mathcal{A}$ is a Ramsey class, then it is necessarily a Fraïssé class, and so has a Fraïssé limit M.

He thus suggested a programme which has not been completed but has proved to be very productive. To find all Ramsey classes, we should determine the homogeneous structures, find their ages, and check the Ramsey property.

In particular, several examples of Fraïssé classes of labelled finite objects that we have seen turn out to be Ramsey classes, including graphs, triangle-free graphs, permutations, and metric spaces.

Topology of automorphism groups

There is a natural topology defined on the symmetric group on a countable set such as $\mathbb{N}$, the topology of pointwise convergence: two permutations g and h are close together they agree on a long initial segment of $\mathbb{N}$. Without going into details, here are a couple of facts about this topology:

- it is derived from a metric, and the group is complete with respect to this metric;
- a subgroup of the symmetric group is closed if and only if it is the automorphism group of something (and "something" can be taken to be a relational structure on the domain of the group).

Thus, automorphism groups are complete metric spaces in their own right.

The Kechris–Pestov–Todorcevic theorem

There are various forms of amenability defined for topological groups. Probably the strongest is "extreme amenability": a group G is extremely amenable if, whenever it acts continuously on a compact Hausdorff space, it has a fixed point.

Examples of this property had been found in the places where topological groups usually live. But the theorem of Kechris, Pestov and Todorcevic was something of a surprise:

Theorem

Let $\mathcal{A}$ be a Fraïssé class of structures with Fraïssé limit M. Then the following are equivalent:

- *$\mathcal{A}$ is a Ramsey class;*
- *the automorphism group of M is extremely amenable.*

An application

We saw that, in a Ramsey class, the structures must be rigid (have trivial automorphism group), and that this can be achieved by including a linear order in the language, so that the structures are labelled.

It turns out that this is the only way to achieve it. For suppose that $\mathcal{A}$ is a Fraïssé class, and M its Fraïssé limit. It is not hard to show that the set of linear orders of M is a compact Hausdorff space. By the KPT theorem, there is a linear order fixed by the automorphism group. This induces a linear order on each structure in the age of M.

The pseudo-arc

What does a typical closed connected subset of the unit square look like?

We have to be careful about the word "typical". In a probability space this can mean "a set of measure 1", but here we don't have a measure. Instead we use a notion from Baire category: in a complete metric space, a set is residual if it contains a countable intersection of open dense subsets. Residual sets behave like complements of null sets: they are non-empty, meet every open set, and any two (or countably many) of them intersect in a residual set.

The metric we use on closed subsets of the square is Hausdorff metric: two sets are within distance ϵ if every point of one is within distance ϵ from some point of the other.

Now it turns out that there is a space $\mathbb{P}$ such that, in the set of closed connected subsets of the unit square with the Hausdorff metric, the elements homeomorphic to $\mathbb{P}$ form a residual set. The space $\mathbb{P}$ is the pseudo-arc.

Several different constructions were given (the first by Knaster in 1922), but R. H. Bing showed that they all produced the same object, and that the homeomorphism group of $\mathbb{P}$ acts transitively on its points.

Moreover, the statement in the first paragraph remains true if we replace the unit square by the unit hypercube in $\mathbb{R}^n$ for any $n \geq 2$, or in Hilbert space.

Its topological definition might suggest that it cannot be constructed by discrete methods, but this is not so ...

Homomorphisms

Our maps so far have always been embeddings $A \to B$, where the image of A is an induced substructure of B (an induced subgraph, in the case of graphs).

For the next example, we need the more general notion of homomorphism. A homomorphism from A to B is a map with the property that, if a relation R is satisfied by a tuple of points in A, then it is satisfied by their images in B.

Thus, for graphs, it is a map taking edges to edges (but not caring about non-edges). So, for example, a homomorphism from G to a complete graph K_r is the same as a proper colouring of the vertices of G with r colours.

As usual, an epimorphism is a homomorphism which is onto. So an epimorphism from G to K_r is a proper colouring which uses all the colours.

Turning the arrows round

I imagine that a category theorist, looking at Fraïssé's theorem, would say, "Form the dual, by turning all the arrows around". This can be done, and has been done by several people. We replace embeddings by epimorphisms, and direct limits by inverse limits. The exact analogue of Fraïssé's theorem holds: it is called the projective Fraïssé theorem. The limit structure P is projectively homogeneous: any two epimorphisms from P to a finite structure are related by an automorphism of P.

There is an additional feature. The inverse limit of a family of finite sets and maps is a subset of the Cartesian product of the finite sets, and so has a topology, induced from the Tychonoff product topology of the discrete topologies on the finite sets. It is not discrete, but is totally disconnected.

Digression: Cantor space

The Cantor space can be regarded as the set of infinite paths from the root in the infinite rooted binary tree. Two paths which first diverge at level n in the tree have distance 2^{-n}. The topology derived from this metric is totally disconnected: the set of paths passing through a fixed vertex is a clopen set, and so clearly any two paths are in disjoint clopen sets.

Now if we code a path by an infinite sequence with 0 for left and 2 for right, we get the standard representation of Cantor space by "ternary decimals".

Putting 0 for left and 1 for right, and using binary decimals, we get a surjection to the unit interval. We can think of this as factoring out an equivalence relation where equivalence classes have size 1 or 2: this turns a totally disconnected space into a connected one.

Constructing the pseudo-arc

Consider the class $\mathcal{P}$ of reflexive paths, graphs which consist of a finite path with a loop at each vertex. Irwin and Solecki show that $\mathcal{P}$ is a projective Fraïssé class, so has a projective Fraïssé limit P.

Thus P has the structure of a graph and the topology of the Cantor set.

They show further that the graph structure on P consists of isolated vertices and edges (with loops) only; thus, an equivalence relation with all equivalence classes of size 1 or 2.

Taking the quotient of P by this equivalence relation gives the pseudo-arc $\mathbb{P}$.

The last step mirrors the step from Cantor space to the unit interval.

Using this, Solecki and Tsankov were able to give a new proof of Bing's theorem that the pseudo-arc has a transitive homeomorphism group.

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for your attention!