# Remembering Donald Preece

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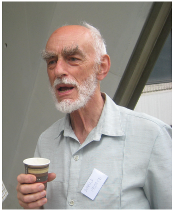
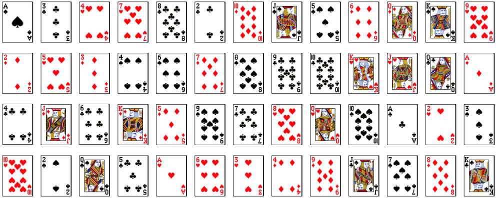
BCC 2015, Warwick





## The design on the title page

Donald found this design while at Rothamsted Experimental Station in the early 1980s. It is a layout of a pack of playing cards in 4 rows of 13 so that

- each value A, 2, . . . , J, Q, K occurs once in each row;
- each suit occurs once in each column;
- any two values occur together in precisely one column (in other words, the columns and values form the projective plane of order 3);
- any row contains four cards of one suit and three of each of the others (in other words, the rows and suits form a trivial 2-$(4, 1, 0)$ design superimposed on three copies of the complete design).

He was so proud of this that he stuck the cards on a backing and put it up on the wall. I show this version (different from the one he published).

# Donald Arthur Preece

- Born 2 October 1939, Edinburgh
- George Heriot's School, Edinburgh
- Associate, Royal College of Organists, 1958
- University of St Andrews, 1958–1962, medals for Pure and Applied Mathematics
- Cambridge Diploma in Mathematical Statistics 1963
- Rothamsted Experimental Station 1963–1969
- University of Kent, West Malling Experimental Station
- Queen Mary, University of London, 2000–2008; emeritus, 2008–2014
- Died 6 January 2014, Edinburgh

An obituary and bibliography by Rosemary Bailey can be found on the arXiv (1402.2220).

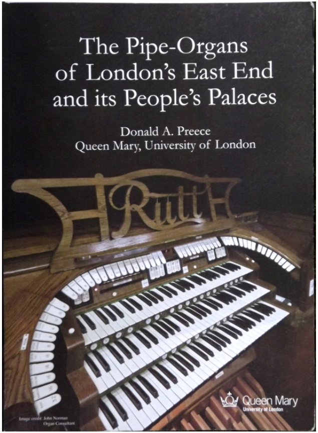
# Donald and the BCC

The 4th BCC (Aberystwyth 1973) perhaps marked the beginning of Donald's move into combinatorics. I sat next to him on the excursion coach, and the result of that discussion was a joint publication constructing some designs resembling the one on the title page of these slides.

Many BCC delegates over the years will know Donald's organisation of the conference concert. The amount of energy he put into this, both physical and nervous, was phenomenal. He would act as accompanist when required for almost anything.

But his biggest contribution occurred in 1999. The committee found itself without a conference venue, due to circumstances beyond our control. Donald stepped in and, with John Lamb's help, organised a very successful BCC at the University of Kent at Canterbury.

# East End organs




One of Donald's big projects at Queen Mary was his survey of pipe organs in the East End of London. At the same time, he was involved in the restoration of the Rutt organ in the Great Hall at Queen Mary, and one of his compositions was played at its opening.

# Primitive lambda-roots

I want to discuss two pieces of work I did with Donald during his time at Queen Mary, University of London, on primitive lambda-roots and on generators in arithmetic progression.
It is very well known to those in the field that primitive roots modulo a prime, or more generally in a finite field, are useful in various combinatorial constructions. But what are we to do if we need a design where the number of points is not prime (a frequent occurrence in statistics)?
I will give one of Donald's constructions which shows how he used ingenuity to bridge the gap. The next slide is in Donald's words.

Consider the following sequence of the elements of $\mathbb{Z}_{35}$:

START

$$\begin{array}{cccccccccccccccccc}
10 & 15 & 5 & 3 & 9 & 27 & 11 & 33 & 29 & 17 & 16 & 13 & 4 & 12 & 1 & 21 & 7 & \searrow \\
 & & & & & & & & & & & & & & & & & 0 \\
25 & 20 & 30 & 32 & 26 & 8 & 24 & 2 & 6 & 18 & 19 & 22 & 31 & 23 & 34 & 14 & 28 & \swarrow
\end{array}$$

FINISH

The last 17 entries, in reverse order, are the negatives of the first 17, which, with the zero, can also be written

$$5^5 \quad 5^6 \quad 5^7 \mid 3^1 \quad 3^2 \quad 3^3 \quad 3^4 \quad 3^5 \quad 3^6 \quad 3^7 \quad 3^8 \quad 3^9 \quad 3^{10} \quad 3^{11} \quad 3^{12} \mid 7^4 \quad 7^5 \mid 0.$$

If we write the respective entries here as $x_i$ ($i = 1, 2, \ldots, 18$), then the successive differences $x_{i+1} - x_i$ ($i = 1, 2, \ldots, 17$) are

$$5 \quad -10 \quad -2 \quad 6 \quad -17 \quad -16 \quad -13 \quad -4 \quad -12 \quad -1 \quad -3 \quad -9 \quad 8 \quad -11 \quad -15 \quad -14 \quad -7.$$

Ignoring minus signs, these differences consist of each of the values $1, 2, \ldots, 17$ exactly once. This is a special type of terrace.

Carmichael's lambda-function $\lambda(n)$ is the maximum order of an element in the group of units of $\mathbb{Z}_n$, the integers mod $n$. An element of this group $U_n$ is a primitive lambda-root if its order is $\lambda(n)$.

Thus, if $n$ is prime, $\lambda(n) = n - 1$ and primitive lambda-roots are just primitive roots.

In the preceding example, $\lambda(35)$ is the least common multiple of $\lambda(5) = 4$ and $\lambda(7) = 6$, that is, $\lambda(35) = 12$. It can be verified that 3 is a primitive lambda-root mod 35.

Motivated by this, Donald and I embarked on a study of primitive lambda-roots. We never found a suitable place to publish it, but you can access the notes (and the GAP functions I wrote for computing with them) at
`https://cameroncounts.wordpress.com/lecture-notes/`
(I should add that I never persuaded Donald to use the computer to do these calculations: he worked on paper on the train journey to London from East Malling.)

The notes are mainly expository, and contain many open problems. There are some unexpected connections. For example, if $\lambda^*(m)$ is the greatest $n$ such that $\lambda(n) = m$, then $\lambda^*(2m)$ is also the denominator of the Bernoulli number $B_{2m}$, re-scaled. We give a proof, but I don't really understand why.

# Generators in AP

This investigation led us on to further exploration of the group $U_n$ of units in $\mathbb{Z}_n$. I guess that Donald had some combinatorial constructions in mind, but I have no idea what they were. As with so many things he did, the work was driven by examples. Here are two. We write

$$U_n = \langle x \rangle_a \times \langle y \rangle_b \times \langle z \rangle_c$$

to denote that $U_n$ is the direct product of cyclic subgroups generated by $x, y, z$, and that the orders of these elements are $a, b, c$ respectively.

$$U_{61} = \langle 9 \rangle_5 \times \langle 11 \rangle_4 \times \langle 13 \rangle_3,$$

where the orders as well as the generators themselves are in arithmetic progression; and

$$U_{455} = \langle 92 \rangle_4 \times \langle 93 \rangle_{12} \times \langle 94 \rangle_6,$$

where the generators are consecutive and the orders are even.

The way we worked was that Donald would arrive at Queen Mary with a new theorem, based on his extensive hand calculations, and it was my job to find a proof of the theorem. I didn't always succeed, and there are *many* open problems in the paper. Here is one case where I did. But even this raises number-theoretic questions such as whether an infinity of such primes exists. (Donald produced long lists by hand.)

## Theorem

*Let $n$ be a prime congruent to 7 or 31 (mod 36), $n > 7$. Suppose that the roots $x_1$ and $x_2$ of $x^2 + 3x + 3 = 0$ in $\mathbb{Z}_n$ have orders $(n-1)/2$ and $n-1$ respectively. Then*

$$U_n = \langle 2x_2 + 3 \rangle_m \times \langle x_2 + 1 \rangle_3 \times \langle -1 \rangle_2,$$

*where $m = (n-1)/6$.*

This and two similar theorems covered all cases of three generators in AP with orders 2, 3 and $(n-1)/6$ when $n$ is prime.

Among the other things we did in the paper were:

- A "lifting" technique that enabled us to use results about primes to study composite $n$.
- Some examples (but not much theory) about the analogous problem in finite fields (we handled orders $11^2$, $11^3$, $19^2$, $19^3$, $23^2$ and $29^2$).
- A couple of isolated examples of 4-term arithmetic progressions of generators: for example,

$$U_{104} = \langle 77 \rangle_2 \times \langle 79 \rangle_2 \times \langle 81 \rangle_3 \times \langle 83 \rangle_4.$$

We remarked that we had been unable to find decompositions with more than four terms; this is an open problem.

## Donald Preece memorial day

- **Thursday 17 September 2015**
- **Queen Mary University of London**, Fogg building and Great Hall
- Speakers include Ian Anderson, Rosemary Bailey, Peter Cameron, Philip Luke, Byron Morgan, J. P. Morgan, Philip Ogden, Martin Ridout
- Organ concert by Alan Wilson, including music by Donald Preece
- Webpage: `http://www.qmul.ac.uk/~pjc/dapday.html`
- Registration: see link on webpage
- Thanks to the mathematics department at QMUL and the British Combinatorial Committee for support.

See you there!