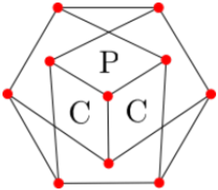
Orbital combinatorics

Peter J. Cameron
Queen Mary University of London
University of St Andrews



Postgraduate Combinatorics Conference
Queen Mary University of London
April 2015

Advertisement

Jack Edmonds will give two courses in London in June:

- At Queen Mary, University of London, 16–19 June: Combinatorial structure of paths, network flows, marriage, routes for Chinese postmen, traveling salesmen, and itinerant preachers, optimum systems of trees and branchings. (Contact: Alex Fink)
- At the London Taught Course Centre, 22–23 June (an LTCC Intensive): Combinatorial structure of submodularity, matroids, learning, transferable & other n-person games, bimatrix games and Nash equilibria. (Contact: Nisha Jones)

Sampling formulae

In how many ways can we sample k objects from a set of size n? As is well known, we have to be more precise about how we sample: with or without replacement, and order significant or not?

	Order significant	Order not significant
With replacement	n^k	$\binom{n+k-1}{k}$
Without replacement	$n(n-1)\cdots(n-k+1)$	$\binom{n}{k}$

Structure v symmetry

The rows of the table on the previous slide tell us something about structure. In the first row there is no restriction, but in the second row the selected items are required to be distinct.
The columns describe how symmetry enters the count. In the first column, we pay no attention to it, but in the second, two selections differing only by a permutation are counted as being the same.
Structure and symmetry are two threads that run through combinatorics (and indeed much of mathematics), and what follows is my attempt to handle both at the same time (in a few special situations).

Chromatic polynomial

The entries in the first column of the table can be interpreted in another way.

The chromatic polynomial of a graph Γ is the polynomial $P_\Gamma(x)$ whose evaluation at a positive integer q gives the number of proper colourings of the vertices with q colours – adjacent vertices must have different colours.

It is a monic polynomial whose degree is the number of vertices of Γ.

So the entries in the first column of the table are evaluations of the chromatic polynomials of the null and complete graphs on k vertices at the value n: the null graph gives no restriction, while the complete graph requires the colours used to be distinct.

The second column counts such colourings up to permutations of the vertices of the graph.

Orbital chromatic polynomial

Our first result combining structure and symmetry is the orbital chromatic polynomial. This is a polynomial $P_{\Gamma,G}(x)$ associated with a graph Γ and a group G of automorphisms of Γ (which may or not be the full automorphism group) with the property that, for a positive integer q, the evaluation $P_{\Gamma,G}(q)$ is the number of orbits of G on propert q-colourings of Γ.
It is a polynomial with degree equal to the number of vertices of Γ and leading coefficient $1/|G|$.
Thus, the second column of our table gives the orbital chromatic polynomials for Γ the null or complete graph on k vertices and G the symmetric group S_k.

To construct the orbital chromatic polynomial, we use the orbit-counting lemma, which asserts that the number of orbits of a finite group G on a finite set X is equal to the average number of fixed points of elements of G.
We take X to be the set of proper q-colourings of Γ, so that $|X| = P_\Gamma(q)$.
Consider an element $g \in G$. If a colouring of Γ is fixed by G, then it is constant on the cycles of G, and so it induces a colouring of the graph Γ/g obtained by shrinking each cycle to a vertex. (If a cycle of g contains an edge of Γ, then there are no fixed colourings, which we can account for by putting a loop on the vertex obtained by contracting this cycle.)
Thus

$$P_{\Gamma,G}(q) = \frac{1}{|G|} \sum_{g \in G} P_{\Gamma/g}(q),$$

giving the required polynomial.

A special case

Consider our earlier example (with a small change in notation), where Γ is the null graph on n vertices and G the symmetric group S_n. Every colouring is proper, and Γ/g is the null graph on the number of vertices equal to the number of cycles of g. So, if $u(n,k)$ is the unsigned Stirling number of the first kind, the number of permutations on n points with k cycles), then

$$\frac{1}{n!}\sum_{k=1}^{n} u(n,k)q^k = \binom{q+n-1}{n}.$$

Multiplying by $n!$, we obtain

$$\sum_{k=1}^{n} u(n,k)q^k = q(q+1)\cdots(q+n-1).$$

Replacing q by $-q$ and multiplying by $(-1)^n$ gives the more familiar formula involving the signed Stirling numbers.

What if names of colours don't matter?

A colouring is a function from vertices to colours. We have discussed symmetries of the graph (as permutations of the vertices). What if permutations of colours are allowed, in other words, we only care about the partition into independent sets (colour classes)?

We can't simply divide by $q!$, where q is the number of colours. This is because some colours may not occur. So first we have to count colourings in which every colour occurs. This is a job for the inclusion-exclusion principle:

$$P^*_\Gamma(q) = \sum_{i=0}^{q} (-1)^{q-i} \binom{q}{i} P_\Gamma(q-i).$$

Then the number of partitions of the graph into q independent sets is just $P_\Gamma(q)/q!$.

Combining?

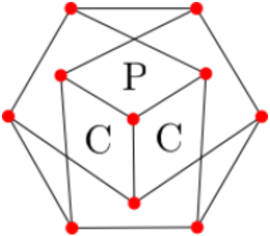
Given a graph Γ and a group G of automorphisms of Γ, can we count the orbits of G on partitions of Γ into q independent sets?

I don't know a "polynomial" method of doing this. An interesting unsolved problem, perhaps!

For graphs which are not too large, it can be done by brute force, by simply computing all such partitions and splitting them into orbits for the group.

The Petersen graph

I will illustrate with the Petersen graph:



As is well known, its automorphism group has order 120 and is isomorphic to the symmetric group S_5.

Chromatic polynomial

For the Petersen graph, the chromatic polynomial is

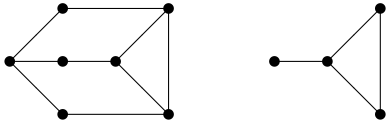
$$P_X(q) = q(q-1)(q-2) \times$$
$$(q^7 - 12q^6 + 67q^5 - 230q^4 + 529q^3 - 814q^2 + 775q - 352).$$

We see that the least number of colors required for a proper coloring (the smallest q for which this is non-zero) is 3, and the number of proper colorings for q up to 10 is given in the following table:

q	3	4	5	6	7	8	9	10
$P_\Gamma(q)$	120	12960	332880	3868080	27767880	144278400	594347040	2055598560

Orbital chromatic polynomial

The only automorphisms whose cycles contain no edges are those corresponding to the identity, transpositions, and 3-cycles in S_5. The graphs X/g for the second and third of these are shown:



We find that the orbital chromatic polynomial is

$$P_{\Gamma,G}(q) = q(q-1)(q-2) \times$$
$$(q^7 - 12q^6 + 67q^5 - 220q^4 + 469q^3 - 664q^2 + 595q - 252)/120.$$

The values for 3 to 10 colors are

q	3	4	5	6	7	8	9	10
$P_{\Gamma,G}(q)$	6	208	3624	36654	248234	1254120	5089392	17449788

Partitions into independent sets

The numbers of such partitions with q parts are now easily found:

q	3	4	5	6	7	8	9	10
$P^*_\Gamma(q)/q!$	20	520	2244	2865	1435	315	30	1

The numbers up to automorphisms of the graph are less easy to compute. Here are the results:

q	3	4	5	6	7	8	9	10
$P^*_{\Gamma,G}(q)$	1	10	30	36	20	7	1	1

Notice how much smaller the numbers are!

Acyclic orientations

An orientation of the edges of a graph Γ is acyclic if there are no directed cycles.
A theorem of Richard Stanley asserts that the number of acyclic orientations of Γ is $(-1)^n P_\Gamma(-1)$, where n is the number of vertices.
Thus, the Petersen graph has 16680 acyclic orientations.
However, it is not true that the number of G-orbits on acyclic orientations is obtained by substituting -1 into the orbital chromatic polynomial.

What we have to do is to define a twisted orbital chromatic polynomial

$$P^{\dagger}_{\Gamma,G}(x) = \frac{1}{|G|} \sum_{g \in G} \sigma(g) P_{\Gamma} / g(x),$$

where σ is the sign of the permutation g; in other words, change the sign of terms coming from odd permutations.
Then the number of G-orbits on acyclic orientations is $(-1)^n P^{\dagger}_{\Gamma,G}(-1)$.
In particular, the number of orbits of acyclic orientations of the Petersen graph is 168.

Beyond chromatic polynomial

There are several possible directions for extending the idea of the orbital chromatic polynomial. Among these are:

- the Tutte polynomial of a graph or matroid, a two-variable generalisation of the chromatic polynomial of a graph;
- the cycle index of a permutation group, a multi-variable polynomial which is useful for a very general class of enumeration problems;
- counting graph homomorphisms, noting that proper colourings of Γ are homomorphisms from Γ to a complete graph.

I will briefly discuss these issues. There is more to be said than I can cover here!

Flows and tensions

In this section, given a graph Γ, we will suppose that we have chosen an orientation of the edges of Γ which is fixed throughout the discussion. Our results will not depend on this orientation.

Let A be a finite abelian group, with identity element 0.

An A-flow on Γ is a function f from directed edges of Γ to A with the property that the signed sum of the values of f on edges at each vertex v (with sign $+$ for edges entering A and $-$ for edges leaving) is 0.

An A-tension on Γ is a function t from directed edges of Γ to A with the property that the signed sum of the values of f around any circuit in Γ (with sign $+$ for edges in the same direction as the circuit, $-$ for edges in the opposite direction) is 0.

A flow or tension is nowhere-zero if it never takes the value 0.

Note that reversing the orientation of an edge and negating the value of the function on that edge preserves the property of being a (nowhere-zero) flow or tension.

There is a close connection between tensions and colourings. The number of nowhere-zero tensions is $q^{-\kappa}P_\Gamma(q)$, where $q = |A|$ and κ is the number of connected components of Γ. In particular, it is independent of the structure of A, depending only on its order.

Bill Tutte showed that the number of nowhere-zero flows is also independent of the structure of A, and is a polynomial in $q = |A|$.

One of the big open questions raised by Tutte is whether every bridgeless graph has a nowhere-zero flow over the cyclic group of order 5.

An orbital version

With Bill Jackson and Jason Rudd, I found orbital versions of the tension and flow polynomials, counting orbits of G on nowhere-zero tensions or flows over A. (The convention is that if a graph automorphism reverses the orientation of an edge, it negates the value of the tension or flow on that edge.)

It turns out that these polynomials are multivariate, with (potentially) countably many variables $x_0, x_1, x_2, \ldots$. To obtain the required orbit numbers, we have to substitute for x_k the number of solutions of $ka = 0$ in the abelian group A. So in general the result does depend on the structure of A.

But only those x_i with $i = 0$ or $i \mid |G|$ occur, which explains why the answer doesn't depend on A if G is the trivial group.

Cycle index

Let G be a permutation group on a set Ω of size n. For each $g \in G$, the cycle index of G is the monomial which records the cycle lengths:

$$z(g) = s_1^{c_1} \cdots s_n^{c_n},$$

where c_i is the number of cycles of length i.
The cycle index of G is obtained by averaging:

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} z(g).$$

This multivariate polynomial, invented by Redfield and Pólya and developed by de Bruijn and others, is useful in many orbit counting problems.

Cycle index theorem

Let F be a set of figures with non-negative integer weights, and let $A(x)$ be the generating function $\sum a_i x^i$, where a_i is the number of figures of weight i. A configuration is a function from Ω to F (think of attaching a figure at each point), and has a weight obtained by summing the weights of all the figures.

Let b_i is the number of orbits of G on functions of weight i, and $B(x) = \sum b_i x^i$.

Theorem

$$B(x) = Z(G)(A(x), A(x^2), \ldots, A(x^n)).$$

Cycle index and Tutte polynomial

For some very special types of group, I was able to produce a polynomial which specialised both to the cycle index of the group and to the Tutte polynomial of an associated matroid. These are the so-called IBIS groups, invented by Dima Fon-Der-Flaass and me.

Bases

A base for a permutation group is a sequence of points of the domain whose pointwise stabiliser is the identity. Bases are important in computational group theory. A base $(x_1, \ldots, x_b)$ is irredundant if no point x_i is fixed by the stabiliser of its predecessors.
Irredundant bases in general lack the nice properties of matroid bases. But Dima and I proved the following theorem:

Theorem

For a permutation group G on Ω, the following conditions are equivalent:

1. *all irredundant bases have the same size;*
2. *the irredundant bases are preserved by re-ordering;*
3. *the irredundant bases are the bases of a matroid.*

A permutation group satisfying these conditions is called an IBIS group (for "Irredundant Bases of Invariant Size").

Graph homomorphisms

My final topic concerns counting graph homomorphisms.
A homomorphism from a graph Γ to a graph Δ is a map from the vertices of Γ to those of Δ which maps edges to edges. (What it does to non-edges is not specified: a non-edge may map to a non-edge, or to an edge, or collapse to a single vertex.)
A homomorphism from Γ to a complete graph K_q with q vertices is just a proper colouring of Γ with q colours. So the counting problem extends the chromatic polynomial, and we would like to have an orbital version.
Homomorphism counting has been studied by Delia Garijo, Andrew Goodall, and Jarik Nešetřil. The orbital version has not been studied …
This topic is also related to the important theory of graph limits developed by László Lovász and various collaborators.

Acting on both sides

We saw in the case of the chromatic polynomial that it is interesting to count orbits on colourings of the automorphism group of the graph, the symmetric group on the set of colours, or the combination of the two groups.
A similar thing happens here. If $F(\Gamma, \Delta)$ denotes the set of homomorphisms from Γ to Δ, then

- the automorphism group of Γ acts on homomorphisms by permuting the arguments: $f^g(v) = f(vg^{-1})$;
- the automorphism group of Δ acts by permuting the values: $f^h(v) = f(v)^h$.

So after counting homomorphisms, we have three orbit-counting problems to solve ...

Graph endomorphisms

An endomorphism is a homomorphism from a graph to itself.
The important new feature here is that, as well as the counting problem, we have algebraic structure: endomorphisms can be composed, and form a monoid (a semigroup with identity), called the endomorphism monoid of the graph.
For some particular graphs, endomorphisms describe well-studied combinatorial problems, and the algebraically defined classes turn out to be orbits of suitable group actions.

An example: Latin squares

The square lattice graph $L_2(n)$ has as vertices an $n \times n$ grid, with two vertices adjacent if they lie in the same row or column. It is the line graph of the complete bipartite graph $K_{n,n}$.
The graph has clique number and chromatic number n: the n-cliques are the rows and columns of the grid.
It can be shown that any endomorphism which is not a homomorphism is an colouring: its image is an n-clique in the graph, and if we number its vertices from 1 to n and then label every vertex of the graph with its preimage, we obtain a Latin square (see next slide).
Every Latin square arises in this way.