Regular polytopes with symmetric and alternating groups

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Algebra Seminar
Lisboa, 25 outubro 2015

Acknowledgement

This is joint work with Maria Elisa Fernandes (Aviero), Dimitri Leemans (Auckland) and Mark Mixer (Boston); the most recent part was done in September–October this year in Aveiro.

Polytopes are objects which have combinatorial, geometric and algebraic aspects.
I will be particularly concerned with regular polytopes, which are generalisations of the classical regular polyhedra in 3-space. They are polytopes which have the maximal amount of symmetry (in a precise sense), and not surprisingly their study has very close connections with group theory.
However, there are many questions here which haven't been very much considered by group theorists.
I begin with something that seems at first glance to have nothing at all to do with polytopes, but there is a connection ...

Independent generating sets

Let G be a finite group. A set $\{g_1, \ldots, g_r\}$ of elements of G is independent if none of the elements lies in the subgroup generated by the others. It is an independent generating set if, in addition, the whole set generates the group G.
Thus independent generating sets resemble bases for vector spaces in elementary linear algebra. However, they do not have the nice properties of bases such as the exchange property, and so they are not the bases of a matroid.

In the symmetric group

Theorem (Julius Whiston, 2000)

The largest size of an independent set in the symmetric group S_n is $n-1$; equality holds if and only if the set is an independent generating set.

In 2002, Philippe Cara and I found all the independent generating sets of size $n-1$ in S_n, for $n \geq 7$. There are two types:

- The first type consists of the transpositions corresponding to the edges of a tree on n vertices.
- The second type contains one transposition; the other elements are 3-cycles and double transpositions. These will not be relevant in what follows.

There are a few extra types for small n. For example, for $n = 6$, we can take images of the above types under the outer automorphism of S_6.

Subgroup lattices

Let $L(G)$ denote the subgroup lattice of the group G.

Proposition

For any finite group G, the Boolean lattice $B(r)$ is embeddable as a meet-semilattice of $L(G)$ if and only if it is embeddable as a join-semilattice of $L(G)$. The largest number r for which these equivalent properties hold is equal to the size of the largest independent subset of G.

If $\{g_1, \ldots, g_r\}$ is an independent set in G, then the subgroups generated by its subsets form a join-semilattice of $L(G)$ isomorphic to $B(r)$.
Note that the above conditions are not equivalent to the embeddability of $B(r)$ in $L(G)$ as a lattice!

Digression: longest chain

A related parameter of the subgroup lattice of a group is the length of the longest chain of subgroups.
This parameter for the symmetric group is of interest in computational group theory (it is an upper bound for base size), and Babai gave an upper bound $2n - 3$ for it.
I found the exact value in 1982; it is published in a paper with Ron Solomon and Alexandre Turull. It is

$$l(S_n) = \left\lceil \frac{3n}{2} \right\rceil - b(n) - 1,$$

where $b(n)$ is the number of 1s in the base 2 representation of n.
Sub-digression: Max Gadouleau, James Mitchell, Yann Peresse and I have similar results for sub-semigroup chains in various important semigroups.

Polytopes

A polytope of dimension r is a generalisation of polygon (in 2 dimensions) or polyhedron (in 3 dimensions) to arbitrary dimension.
It can be regarded as a partially ordered set (the elements are the faces of various dimensions) in which all maximal chains contain $r + 2$ elements (including a bottom element $\emptyset$ of dimension -1 and a top element of dimension r which represents the whole polytope). Each element can be assigned a unique dimension, corresponding to the position it occupies in a maximal chain. Elements of dimenion 0, 1, 2 are vertices, edges, and faces.
The maximal chains are called flags.
We require several further conditions (see next slide).

- For $i < j < k$, if x, y, z are elements of dimensions i, j, k with $x \leq y$ and $y \leq z$, then $x \leq z$.
- If x and y have dimensions i and $i + 2$ and $x < y$, then there are just two elements z satisfying $x < z < y$.
- A strong connectedness condition: if F and G are two flags, then there is a sequence of flags beginning at F and ending at G, such that consecutive members intersect in all but one of their elements, and that $F \cap G$ is contained in every flag in the sequence.

The poset obtained by reversing the order is also a polytope, called the dual of the original.
If x and y are elements of a polytope with $x < y$, then the interval $[x, y] = \{z : x \leq z \leq y\}$ is itself a polytope, of dimension $\dim(y) - \dim(x) - 2$. In particular, if $\dim(y) - \dim(x) = 3$, then $[x, y]$ is a polygon.

Regular polytopes

If two flags $(x_{-1}, x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_r)$ and $(x_{-1}, x_0, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_r)$ differ only in the element of dimension i, then any automorphism which fixes the first flag also fixes the second.

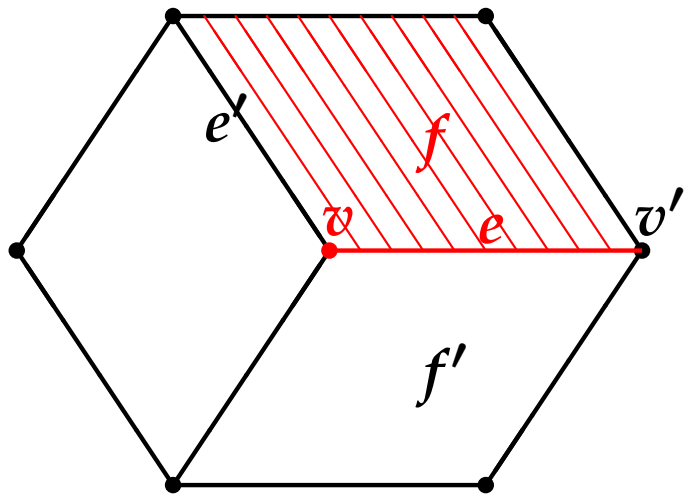
Hence, using the strong connectedness property, any automorphism which fixes a flag must fix every flag, and hence is the identity.

A polytope is regular if the automorphism group acts transitively on the flags. In this situation, the action of the group is regular: there is a bijection between flags and automorphisms. (We fix a reference flag F, and then identify F' with the unique automorphism mapping F to F'.)

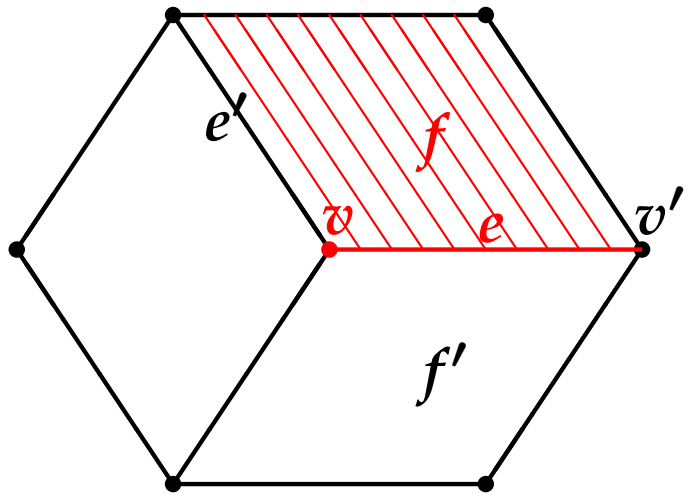
If a polytope is regular, then for any i, if $\dim(x) = i - 1$, $\dim(y) = i + 2$, and $x < y$, then $[x, y]$ is a p_i-gon, where p_i depends on i but not on x and y. The vector $(p_0, p_1, \ldots, p_{r-1})$ is the Schläfli symbol of the polytope.

String C-groups

Because of the correspondence between the set of flags and the automorphism group G of a polytope, it is possible to translate everything into the group. We will see that the existence of a regular polytope is equivalent to a sequence of group elements with certain properties.
To motivate this, consider the cube.



Our reference flag is $(\emptyset, v, e, f, C)$ (where C denotes the cube). Let s_v, s_e and s_f be the automorphisms mapping it to $(\emptyset, v', e, f, C)$, $(\emptyset, v, e', f, C)$ and $(\emptyset, v, e, f', C)$ respectively.



Now s_v maps v' back to v, and so $s_v^2 = 1$; similarly $s_e^2 = s_f^2 = 1$. Also $s_v s_e$ rotates the square face f one step clockwise, and so $(s_v s_e)^4 = 1$. Similarly $(s_e s_f)^3 = 1$. And s_v and s_f both fix e, and so they commute: $(s_v s_f)^2 = 1$.

More generally, we define a string C-group to be a finite group generated by elements $s_0, s_1, \ldots, s_{r-1}$ satisfying the conditions

- $s_i^2 = 1$.
- s_i and s_j commute if $|i - j| > 1$ (the string condition).
- For $I \subseteq \{0, \ldots, r-1\}$, let S_I denote the subgroup generated by $\{s_i : i \in I\}$. Then $S_I \cap S_J = S_{I \cap J}$ for any I and J (the intersection condition).

Theorem

The existence of a regular polytope with automorphism group G is "equivalent" (in a suitable sense) to a representation of G as a string C-group.

Note that the order of $s_i s_{i+1}$ is the ith component of the Schläfli symbol of the polytope.

We do not insist that s_i and s_j fail to commute if $|i - j| > 1$. In other words, we allow degenerate polytopes where some of the polygons are digons. This might seem to make things harder, but actually makes them much easier. The subgroup generated by a subset of $\{s_0, \ldots, s_{r-1}\}$ is a string C-group in its own right, so we have the possibility of induction!

Also, we do not assume that the orders of the s_i and $s_i s_j$ give a presentation of a group. (If they do, then the group is a Coxeter group.)

Finally, the intersection condition shows that $\{s_0, \ldots, s_{r-1}\}$ is an independent generating set for G. Indeed, it is stronger: it is equivalent to the condition that the map $I \mapsto G_I$ embeds the Boolean lattice $B(r)$ as a sublattice of the subgroup lattice $L(G)$ of G.

The symmetric group, 1

It follows from Whiston's theorem that the dimension of a polytope with autmorphism group S_n is at most $n-1$. It further follows from the theorem of Cameron and Cara that there is a unique such polytope of dimension $n-1$. (The condition that generators are involutions rules out the second type; the string condition shows that the tree is a string.) The generators are $s_i = (i+1, i+2)$ for $i = 0, \ldots, n-2$.
The corresponding polytope is the regular $(n-1)$-simplex, whose faces are all the subsets of $\{1, \ldots, n\}$.

The symmetric group, 2

Fernandes, Leemans and Mixer asked about regular polytopes of smaller dimension r with group S_n. They computed the following table:

$n \backslash r$	3	4	5	6	7	8	9	10	11	12	13
5	4	1	0	0	0	0	0	0	0	0	0
6	2	4	1	0	0	0	0	0	0	0	0
7	35	7	1	1	0	0	0	0	0	0	0
8	68	36	11	1	1	0	0	0	0	0	0
9	129	37	7	7	1	1	0	0	0	0	0
10	413	203	52	13	7	1	1	0	0	0	0
11	1221	189	43	25	9	7	1	1	0	0	0
12	3346	940	183	75	40	9	7	1	1	0	0
13	7163	863	171	123	41	35	9	7	1	1	0
14	23126	3945	978	303	163	54	35	9	7	1	1

We see the entries 1 for $r = n - 1$ corresponding to the regular simplices, and we have seen that there are no more. Note also the entries 1 for $r = n - 2, n \geq 7$; 7 for $r = n - 3, n \geq 9$; 9 for $r = n - 4, n \geq 11$; and 35 for $r = n - 5, n \geq 13$.
This suggests the conjecture:

Conjecture

Given k, there is a number $N(k)$ such that, for $n \geq 2k + 3$, the number of regular polytopes of dimension $n - k$ with automorphism group S_n is $N(k)$.

Fernandes, Leemans and Mixer have established this conjecture for $k \leq 4$, with the values of $N(k)$ given above.

The alternating groups

We saw that regular polytopes with a given group (like S_n) can be studied by induction, using the fact that any subset of the generators of the string C-group themselves generate a smaller string C-group.

Fernandes, Leemans and Mixer examined the alternating group A_n. They conjectured that the largest dimension of a regular polytope with group A_n is $\lfloor (n-1)/2 \rfloor$ for $n > 11$. (There is an exceptional example of rank 6 for A_{11}.) They managed to construct examples meeting the conjectured bound.

This, incidentally, shows that there is a big difference between largest dimension of a polytope with group G, and largest independent generating set for G (which is $n-2$ for $G = A_n$).

A theorem

At the end of a month's very hard work in Aveiro (building on what the other three had done over several years), we believe we have proved the conjecture:

Theorem

For $n > 11$, the largest rank of a regular polytope with automorphism group A_n is $\lfloor (n-1)/2 \rfloor$.

We are not yet ready to say for sure that we have a proof: the paper will be nearly 50 pages long, and we finished going through it at 16:45 on Friday 16 October; Dimitri and I left Aveiro at the weekend. There are some small gaps which we believe we know how to fill.

Some words about the proof

Let Γ be a string C-group with generators $\{\rho_s : s \in S\}$ which is isomorphic to the alternating group A_n. For $i \in S$, let $\Gamma_i = \langle \rho_s : s \neq i \rangle$.
We divide the analysis into three cases:

- some Γ_i is primitive (in its action in $\{1, \ldots, n\}$);
- some Γ_i is transitive imprimitive;
- all Γ_i are intransitive.

I will say a few words about each case. Note that several small cases are handled by computer.

Γ_i primitive

For primitive groups, it follows from CFSG that they are small: with "known" exceptions, they have order at most $n^{1+\log_2 n}$.
(The most precise form of this result is due to Attila Maróti.)
On the other hand, a string C-group of rank r clearly has order at least 2^r; and if the diagram is connected, Marston Conder improved this lower bound to a best-possible result $4^r/2$ for $n \geq 9$.
A small amount of further trickery gives the result in this case.

Γ_i transitive imprimitive

Choose maximal blocks for Γ_i, so that the action on the blocks is primitive. Suppose that there are m blocks, each of size k.
Let L be a subset of S which forms an independent generating set for the action on blocks; C the set of elements of S commuting with every element of L; and R the remainder of S.
Then

- $|L| \leq m - 1$;
- $\langle C \rangle$ acts in the same way on each block, so $|C| \leq k - 1$;
- either L is disconnected or $|R| \leq 2$.

If L is disconnected then the primitive group on blocks is a direct product, so $|L| \leq 2 \log_2 m$. Otherwise we have $r \leq k + m + 1$, which gives the required result unless $k = 2$ or $m = 2$ or finitely many others. These cases require special treatment.

All Γ_i intransitive

In this case, each generator ρ_i interchanges points in different Γ_i-orbits. We construct an edge-labelled graph with label set S (called a fracture graph) by choosing one such transposition for each i (labelled i).
The rest of the argument (by far the longest part) involves careful analysis of the fracture graphs.
The easier case is when, for each i, there are at least two ρ_i-edges joining points in different Γ_i-orbits. In this case we construct a 2-fracture graph by choosing two such edges. We show it is possible to choose such a graph so that one component is a tree and all others have at most one cycle. Then $2r$ (the number of edges) does not exceed $n-1$, and we have the result.