Permutation pattern classes as ages of relational structures

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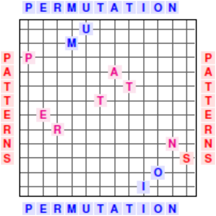


Permutations and substitutions

In Galois' time, the word *permutation* meant an arrangement, typically of the first n natural numbers; the word *substitution* meant the operation of rearranging these numbers, that is, the bijective function from $\{1, \ldots, n\}$ to itself mapping i to the ith number in the arrangement.

Now the word "substitution" has been lost, and "permutation" has to do duty in both senses.

So, when Galois wrote about *groupe de substitutions*, he means what we now call a *permutation group*.

We sometimes distinguish the two senses by talking about the "active" and "passive" forms of a permutation. Typically permutation group theory takes the first form, and permutation pattern theory the second.

Permutations as relational structures

Everyone here understands the order relation on permutations: for example, $132 \preccurlyeq 2413$. A permutation pattern class is a set $\mathcal{C}$ of permutations with the property that, if $\pi \in \mathcal{C}$ and $\sigma \preccurlyeq \pi$, then $\sigma \in \mathcal{C}$.

There is actually a way to regard a permutation as a mathematical structure similar to a graph, tournament, poset, etc., so that the relation of "subpattern" is the exact analogue of, say, "induced subgraph".

A permutation is a set X with a pair $(<_1, <_2)$ of total orders. Assuming for the moment that X is finite, each order $<_i$ establishes a bijection between X and $\{1, \ldots, n\}$, where $n = |X|$. The composition of the inverse of the first bijection with the second gives the permutation in the usual sense.

Thus, the permutation 2413 is represented by taking $X = \{a, b, c, d\}$, with $a <_1 b <_1 c <_1 d$ and $b <_2 d <_2 a <_2 c$. The involvement $132 \preccurlyeq 2413$ is seen by considering the induced substructure on $\{b, c, d\}$.

Nothing new

This is actually common practice already. Permutations arise when the labels in a labelled structure can be read in two different ways:

- Given a word with no repeated letters, we can order the letters alphabetically or in the order they occur: for example GALOIS gives us the permutation 214536.
- The vertices of a labelled tree can be read in depth-first or breadth-first order.
- Computational devices which re-order strings: the terms have an input and an output order.
- We saw a related example in Monday's talk by Manda Riehl.

More generally ...

Now there is a natural way to generalise permutations to what might be called multipermutations. One of these is just a set carrying a k-tuple of total orders.

We can ask much the same questions about multipermutations as we do about permutations. But maybe we shouldn't be too surprised if these objects don't have ready-made applications the way permutations do ...

The conference logo really exemplifies the case $k = 3$, since the letters are ordered alphabetically as well as in the two words of the title.

Relational structures

The context for these objects is the theory of relational structures, or relations for short (as Roland Fraïssé called them in his influential book *Theory of Relations*.

A relational structure S is just a first-order structure over a language containing no constant or function symbols. In other words, it consists of a set Ω carrying a collection of (named) relations. Graphs, directed graphs, posets are examples, and with the above interpretation also permutations and multipermutations.

Ages

Unlike common practice in logic, I will allow the empty set to carry a relational structure (but I forbid nullary relations, so there is only one structure on the empty set).
A substructure of S will always be an induced substructure; in other words, given a subset, we take all instances of the relations within that subset.
Fraïssé introduced some idiosyncratic terminology for studying these structures. The age of a relational structure S is the class $\mathrm{Age}(S)$ of all finite relational structures (over the same language) which are embeddable in S.

Recognising an age

Two obvious necessary conditions for a class $\mathcal{C}$ of finite relational structures to be an age are:

(A1) $\mathcal{C}$ is closed under isomorphism;

(A2) $\mathcal{C}$ is closed under taking substructures.

I will always assume that structures are no larger than countably infinite. With this assumption, we also have:

(A3) $\mathcal{C}$ has only countably many members (up to isomorphism).

For example, the class of finite metric spaces is not the age of any countable metric spaces, since there are uncountably many two-point metric spaces. (It is however the age of the celebrated Urysohn space: but that is another story!)

The joint embedding property

There is another, less obvious condition, satisfied by an age, the joint embedding property or JEP:

(A4) For any $B_1, B_2 \in \mathcal{C}$, there exists $C \in \mathcal{C}$ such that B_1 and B_2 are both embeddable in $\mathcal{C}$.

Theorem
A class of finite relational structures is the age of some countable structure if and only if it has properties (A1)–(A4).

The proof is just what Wilfrid Hodges described as "shovelling everything in".
The countable structure is usually far from unique. There are many countable graphs which embed all finite graphs.

Permutations

As you will have recognised, a permutation pattern class is a set of finite permutations satisfying (A1) and (A2). (We can blur the distinction between "class" and "set", since any class satisfying (A1) has a unique member of each isomorphism class for which the domain is $\{1, \ldots, n\}$ and $<_1$ is the natural order. Note also that (A3) is vacuous for permutation pattern classes.) The case for permutations of the above theorem is Theorem 1.2 of the paper of Atkinson, Murphy and Ruškuc in 2005. They give four equivalent conditions of which condition 3 is the JEP. These authors call a permutation pattern class satisfying the JEP an atomic class, since it cannot be broken into two subclasses both satisfying (A1) and (A2).

The amalgamation property

Fraïssé proposed the following strengthening of the JEP, known as the amalgamation property, or AP.

(A5) Given $A, B_1, B_2 \in \mathcal{C}$, with embeddings $f_i : A \to B_i$ for $i = 1, 2$, there exists $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ such that $f_1 g_1 = f_2 g_2$.

This means that two structures in $\mathcal{C}$ containing isomorphic substructures can be "glued together" along the isomorphism inside a structure in $\mathcal{C}$.

Our convention about the empty set means that the AP implies the JEP; but be warned, not everybody does this, and often the JEP is needed as an extra condition.

What does the AP do for us?

Homogeneous structures

A relational structure S on Ω is homogeneous if any isomorphism between finite substructures of S can be extended to an automorphism of S.
This is a very strong symmetry condition. Fraïssé proved:

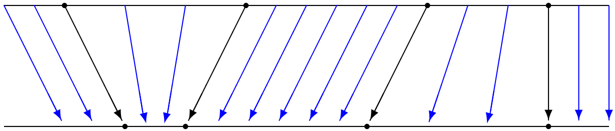
Theorem

A class $\mathcal{C}$ of finite relational structures is the age of a countable homogeneous relational structure S if and only if it satisfies (A1)–(A3) and (A5).
Moreover, if these conditions hold, then S is unique up to isomorphism.

A class satisfying (A1)–(A3) and (A5) is called a Fraïssé class, and the unique countable homogeneous structure S of which it is the age is its Fraïssé limit.

Examples

The most famous example of a countable homogeneous structure is $(\mathbb{Q}, <)$, the rationals as ordered set. This structure was famously characterised by Cantor as the unique countable dense ordered set without endpoints. (These conditions are special cases of the the AP.)
The homogeneity is shown as follows. Given two finite subsets of $\mathbb{Q}$, the unique order-isomorphism between them is given by a a piecewise-linear map which translates the two ends of the interval.



$(\mathbb{Q}, <)$ is the only countable homogeneous total order, and so the class of all finite total orders is the only Fraïssé class.

Permutations

The class of all finite permutations is a Fraïssé class, and its Fraïssé limit is a very interesting object, which I will call the generic permutation.
It is a countable set carrying two total orders, both isomorphic to the order on $\mathbb{Q}$, with the property that any interval in one of the orders, no matter how short, is dense in the other order.
Unfortunately it is the only interesting homogeneous permutation:

Theorem

There are just five homogeneous countable permutations: the increasing permutation, the decreasing permutation, the increasing sequence of decreasing permutations, the decreasing sequence of increasing permutations, and the generic permutation.

Other interesting examples?

There are two places we might look for further nice examples. The class of all k-tuples of finite total orders is a Fraïssé class, and so there is a generic countable k-tuple of total orders. Are there any others?

Problem

Determine the homogeneous countable k-tuples of total orders for $k > 2$.

Greg Cherlin and his student Sam Braunfeld are working on this. They have shown that the class is quite rich: any distributive lattice can occur as the lattice of definable equivalence relations in some homogeneous structure of this kind.

The other possible source is almost-homogeneous structures, which I will describe later.

Ramsey classes

A class of relational structures is a Ramsey class if Ramsey's theorem holds in the class.

We use the notation $\binom{B}{A}$ for the class of all substructures of B isomorphic to A. Now $\mathcal{C}$ is a Ramsey class if, given a natural number r and a pair $A, B \in \mathcal{C}$, there exists $C \in \mathcal{C}$ such that, if the elements of $\binom{C}{A}$ are coloured arbitrarily with r colours, there is an element B' of $\binom{C}{B}$ such that $\binom{B'}{A}$ is monochromatic.

Nešetřil observed:

Theorem

A hereditary isomorphism-closed Ramsey class is a Fraïssé class.

The consequence was that, to find examples of Ramsey classes, we should look at ages of homogeneous structures.

It is not difficult to see that a non-trivial Ramsey class has the property that its objects are rigid. The easiest way to ensure this is to require that one of the relations is a total order.
Now permutation patterns (and, more generally, multiorders) are obvious candidates. It was shown by Böttcher and Foniok (for permutation patterns) and Sokić (for arbitrary multiorders) that

Theorem

The class of all finite multiorders is a Ramsey class.

Topological dynamics

An unexpected addition to this theory came from Kechris, Pestov and Todorcevic.

The symmetric group on an infinite set is a topological group: the basic open sets are the cosets of stabilisers of finite tuples.

For the symmetric group of countable degree (say, acting on $\mathbb{N}$), the topology is derived from a metric: two permutations are close together if they agree on long initial subsequences of $\mathbb{N}$.

Now it can be shown that a subgroup of the symmetric group of countable degree is closed in this topology if and only if it is the full automorphism group of some relational structure (which can be taken to be homogeneous).

A flow for a topological group G is a continuous action of G on a compact space M. A flow is minimal if there is no non-empty proper closed G-invariant subset of M. A mimimal flow is universal if it can be mapped onto any minimal flow by a G-equivariant continuous map.
Now the main definition: a topological group G is extremely amenable if its universal minimal flow consists of a single point. In other words, any continuous action of G on a compact space has a fixed point.
Kechris, Pestov, and Todorcevic came up with a rather unexpected and rich collection of extremely amenable groups:

Theorem

Let X be a countable set, and G a closed subgroup of the symmetric group on X. Then G is extremely amenable if and only if it is the automorphism group of a homogeneous structure whose age is a Ramsey class of ordered structures.

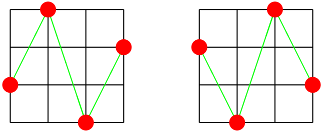
In particular, the automorphism group of the generic permutation pattern (or, more generally, the generic k-order for any k) is extremely amenable.

Almost homogeneous structures

Given any countable structure, it is always possible to add relations to the language to make the structure homogeneous. In interesting cases, these relations are definable (without parameters) in the original structure, and only finitely many of them are needed. We call a structure with this property almost homogeneous.

Here is an example. There is a "generic" bipartite graph, which is not homogeneous (as a graph) since non-adjacent vertices may be at distance 2 or 3. However, if we add the equivalence relation whose equivalence classes form the bipartition, the structure becomes homogeneous; this relation is definable (by the rule that two vertices are equivalent if they are equal or at distance 2 in the graph).

An example



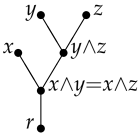
I will call a permutation N-free if it contains neither 2413 nor 3142. The class of N-free permutations is the age of an almost homogeneous structure, as follows.

Let T be a finite binary tree with root r. Let c be a colouring of the internal vertices with two colours, black and white. Let X be the set of leaves of T (excluding r).

From the data (T, r, c) we construct a permutation as follows.

Let $<_1$ be the relation on the leaves defined in the usual way by depth-first search in T, and $<_2$ the order defined by modified depth-first search in which the children of a white vertex are visited in reverse order.

Now define a ternary relation by $x|yz$ if $x \wedge y = x \wedge z \neq y \wedge z$, where $x \wedge y$ is the point where the three paths xy, xr and yr meet.



$x|yz$

The structure of the data (T, r, c) is reflected faithfully on the set X of leaves with its two orders and one ternary relation.
There is a countable permutation whose age consists of all these finite permutations, which is "homogenised" by this ternary relation.

Reducts

A reduct of a first-order structure S is a relational structure (possibly in a different first-order language) whose relations have first-order definitions in S without parameters.
For example, among the reducts of $(\mathbb{Q}, <)$ are:

- the betweenness relation β, with $\beta(x, y, z)$ if $x < y < z$ or $z < y < x$;
- the circular order γ, with $\gamma(x, y, z)$ if $x < y < z$ or $y < z < x$ or $z < x < y$ (think of bending the line into a circle);
- the separation relation σ, so that $\sigma(w, x, y, z)$ holds if the pair $\{x, z\}$ separates y and w.

Note that the automorphism group of a reduct contains the automorphism group of S. In the countable homogeneous case, this property characterises reducts. We identify reducts with the same group.

Reducts of $(\mathbb{Q}, <)$

The first countable homogeneous structure whose reducts were classified was $\mathbb{Q}, <)$. The theorem was not phrased in these terms. Since an overgroup of $\mathrm{Aut}(\mathbb{Q}, <)$ is highly set-transitive (that is, transitive on the set of k-subsets for all finite k), the result was phrased as a classification of the highly set-transitive groups.

Theorem

There are five reducts of $(\mathbb{Q}, <)$: these are $(\mathbb{Q}, <)$, $(\mathbb{Q}, \beta)$, $(\mathbb{Q}, \gamma)$, $(\mathbb{Q}, \sigma)$, and the set $\mathbb{Q}$ with no structure.

Reducts of the generic permutation

As far as I know, the problem of finding the reducts of the generic countable permutation has not been solved. Among them are some of interest in other fields:

- the generic permutation graph, whose edges are the pairs of points on which the two orders disagree (we saw this in David Bevan's talk on Wednesday);
- the generic 2-dimensional poset, the intersection of the two orders.

I know of 37 reducts. Of these, 25 are found by taking reducts of the two orders independently; the other 12 allow the orders to be interchanged, and include the two examples mentioned above.

Problem

Are there any more? And what about the generic k-tuple of orders?

Any reduct might potentially be useful, or might pose interesting questions.

A construction of the generic permutation

The generic permutation is a beautiful but somewhat elusive object. Here is an explicit construction.
We take the elements of the set Ω to be the integer lattice $\mathbb{Z}^3$.
If $\alpha = (a, b, c)$ is a vector whose components are linearly independent over $\mathbb{Q}$, we can associate an order $<$ on Ω by the rule that $v_1 < v_2$ if and only if $v_1.\alpha < v_2.\alpha$. The condition on α guarantees that this is a total order.
If we take two such vectors α and β which are linearly independent, then the two orders so defined form the generic permutation. It is not hard to construct two such vectors.

Cayley objects for groups

We see this construction that the generic permutation admits a transitive abelian group isomorphic to $\mathbb{Z}^3$ (the translation group of the lattice).

Which structures admit transitive abelian groups, or more generally, transitive groups in which the stabiliser of a point is the identity? This class includes the important class of Cayley graphs.

More generally, we say that S is a Cayley object for a group G if G acts regularly on S (transitively with trivial point stabiliser) as a group of automorphisms.

So the question is: For what groups is $(\mathbb{Q}, <)$ a Cayley object?

Right-orderable groups

A group G acts regularly on an ordered set if and only if G is right-orderable: there is an order on G invariant under right translation. Many groups admit such orders.

If G is right-ordered, the set P of positive elements (greater than the identity) satisfies $G = P \cup \{1\} \cup P^{-1}$ (disjoint union) and $P^2 \subseteq P$.

The order is dense (and so isomorphic to $(\mathbb{Q}, <)$, if G is countable) if and only if $P^2 = P$. This happens in most examples.

There are many examples of right-orderable groups, including all non-abelian free groups.

What about permutations, or more general multiorders?

Multiorders

Theorem

The generic k-tuple of orders admits the group $\mathbb{Z}^r$ acting transitively if and only if $r > k$.

The proof involves some ideas from Diophantine approximation such as Kronecker's theorem.
But these are the only finitely generated groups known to act regularly on generic multiorders!

Problem

Are there any others? For example, can free groups act in this way?

And finally . . .

There are other structures where the same game can be played. For example, if you are interested in the subword order on words over a finite alphabet A, then proceed as follows:

- A word is a finite set carrying a total order and also a partition into parts labelled by A;
- A subword is an induced substructure on a subset.

The class of all finite words is a Fraïssé class, so there is a universal homogeneous countable word, whose order is isomorphic to $\mathbb{Q}$.

So, what can we do with this approach? Over to you! And thank you for your attention.