Road colouring and road closures: synchronization and idempotent generation

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Groups & their Applications
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So, at least to get things started, it is natural to consider the case $M = \langle G, a \rangle$, where G is a permutation group and a a non-permutation. A typical question is:

Question

Which permutation groups G guarantee that $M = \langle G, a \rangle$ has some specified nice property, for all or some choices of non-permutation a?

Idempotent generation

Here is a special case. The rank of a map is the size of its image; and an itempotent is a map e satisfying $e^2 = e$.

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For which transitive permutation groups G is it true that, for all maps a of rank 2, the semigroup $\langle G, a\rangle \setminus G$ is idempotent-generated?

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So a necessary and sufficient condition is that G has the k-universal transversal property: given any k-set S and k-partition P, there is an element of G mapping S to a section for P.

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An old theorem of Donald Higman says that this is equivalent to primitivity of the group G, the property that G preserves no non-trivial partitions of $\{1, \ldots, n\}$.

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Let P and S be the kernel and image of a. If there is a product of idempotents in $\langle a, G \rangle \setminus G$ having kernel P' and image S', then the image of each idempotent is a section for the kernel of the next, so there is a path from P' to S' in $H(G, k, P, S)$.

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So connectedness of the Houghton graph is a necessary condition for idempotent generation.

Theorem

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We will say that G has the 2-Hc property if this condition holds. As this theorem suggests, 2-Hc is a strengthening of primitivity.

A reformulation

The condition in the theorem is still time-consuming to check, since there are exponentially many 2-partitions of $\{1, \ldots, n\}$.

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A primitive permutation group G on $\{1, \ldots, n\}$ has the 2-Hc property if and only if, for every G-orbit O on 2-subsets of $\{1, \ldots, n\}$, and every maximal block of imprimitivity B for the action of G on O, the graph with edge set $O \setminus B$ is connected.

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Of course, there are only polynomially many orbital graphs to check. For each one, there are hopefully not too many maximal blocks of imprimitivity. And testing connectedness is fast! So you could just go to the computer, start up GAP, and begin testing examples …

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Of course we have more information: our group is a primitive group acting on the edges of some orbital graph …

An example

Consider the automorphism group of a $m \times m$ grid: two points are joined if they lie in the same row or column. The automorphism group is the wreath product $S_m \wr S_2$ in its product action on m^2 points.

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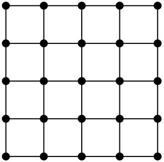

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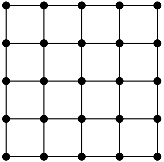

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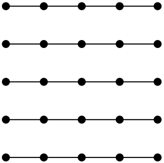

The edges fall into two blocks of imprimitivity under the automorphism group: horizontal and vertical.
If workmen come and dig up all the vertical roads, then it is impossible to get from one row to another. So this primitive group fails to have the 2-Hc property.

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The O'Nan–Scott theorem gives us good information about the basic primitive groups: they must be affine, diagonal, or almost simple.

Second generalisation

Another way of looking at the example leads to the following.

Proposition

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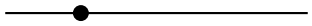
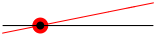
Let G be a primitive permutation group. Suppose that G has an imprimitive subgroup of index 2*. Then G does not have the* 2*-Hc property.*

Groups of this type are groups of automorphisms and anti-automorphisms of self-dual incidence structures, acting on the set of flags (incident point-block pairs) of the structure. We join two flags if they share a point or a block. The automorphisms form a subgroup of index 2, and the edges fall into two blocks depending on whether the shared element is a point or a block. If we remove edges of one type, we cannot move between flags with different elements of the other type.

So there are two kinds of adjacency:

 is joined to 

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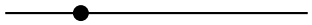
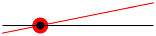
 is joined to 

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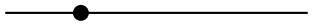
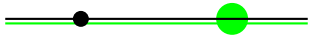
 is joined to 

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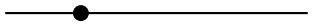
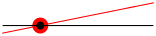
 is joined to 

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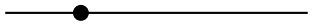
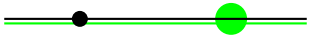
 is joined to 

If all connections of the second type are removed, then we cannot move from a flag to another flag with a different point!

Examples for the last result include groups of projective spaces (on point-hyperplane flags or on point-hyerplane antiflags, or on i-space/$(n-1-i)$-space flags), symplectic generalised quadrangles in characteristic 2, G_2 generalised hexagons in characteristic 3, and some sporadic examples such as $\mathrm{PGL}(2, 11)$ with degree 55 or 66, and $\mathrm{HS} : 2$ with degree 22176 coming from symmetric 2-designs with 2-transitive groups.

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- $L_3(2):2$, degrees 21 and 28 (flags and antiflags in Fano plane);
- $S_6:2$ and subgroups, degree 45;
- $L_3(3):2$, degrees 52 and 117;
- $L_2(11):2$, degrees 55 and 66;
- $\mathrm{Aut}(L_3(4))$ and subgroups, degree 105;
- $S_8 = L_4(2):2$, degrees 105 and 120;
- S_7, degree 120.

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Let p be a prime congruent to $\pm 1 \pmod 5$ and to $\pm 3 \pmod 8$. Then $\mathrm{PGL}(2, p)$ contains a conjugacy class of subgroups isomorphic to A_5, which splits into two classes in $\mathrm{PSL}(2, p)$. An A_4 subgroup of one of these A_5's is normalised by S_4 in $\mathrm{PGL}(2, p)$; elements of S_4 not in A_4 conjugate the A_5 to one in the other $\mathrm{PSL}(2, p)$ class.

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I do not see the prospect of determining all these groups …

From duality to triality

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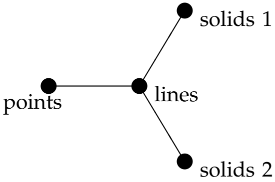

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Triality was discovered by Eduard Study and developed by Élie Cartan. It is connected with other remarkable things such as the octonions, spinors, and the Leech lattice.

A conjecture

Conjecture

Let G be a basic primitive permutation group. Suppose that G does not have an imprimitive normal subgroup of index 2, and is not one of the triality examples just mentioned. Then G has the 2-Hc property. Hence, for any rank 2 map a, the semigroup $\langle G, a\rangle \setminus G$ is idempotent-generated.

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Conjecture

Let G be a basic primitive permutation group. Suppose that G does not have an imprimitive normal subgroup of index 2, and is not one of the triality examples just mentioned. Then G has the 2-Hc property. Hence, for any rank 2 map a, the semigroup $\langle G, a\rangle \setminus G$ is idempotent-generated.

This conjecture has been checked computationally for all degrees up to 130 and many larger degrees. No counterexamples have been found.

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For example, a theorem of Wielandt asserts that a group of degree p^2 (for p prime) is affine, or contained in $S_p \wr S_2$, or is 2-transitive. In the second case, 2-Hc fails, while in the third, it holds. So it is the affine case which has to be considered.

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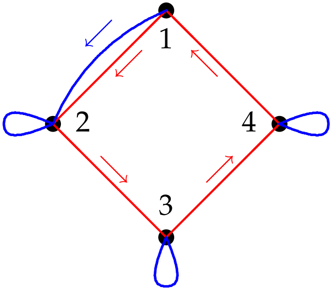

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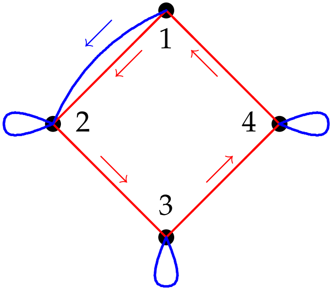

You can check that the sequence BRRRBRRRB will bring you to room 2 from any starting point.

Automata

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Algebraically, a transition is a transformation on the set of states; since we may compose transitions, an automaton is a transformation monoid on the set of states, with a prescribed set of generators.

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We can test whether an automaton is synchronizing in polynomial time, but finding the shortest reset word is NP-hard.

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The road-colouring conjecture (now theorem!) asserts that these necessary conditions are sufficient.

Synchronizing groups

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Our question now is:

Question

Which permutation groups are synchronizing?

This turns out to include many problems of great interest from extremal combinatorics and finite geometry.

Synchronizing groups, 2

Proposition

A synchronizing group is primitive and basic.

For example, if G is not basic, choose a system of imprimitivity, and choose representatives for its blocks: let a map any point to its representative. Then $\langle G, a \rangle$ is not synchronizing.

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It was conjectured for a time that every primitive group is almost synchronizing. But …

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We found two nice counterexamples to the conjecture. These are the automorphism groups of the Tutte–Coxeter graph on 30 vertices and the Biggs–Smith graph on 102 vertices, acting on the edges of the graph.

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Subsequently, more counterexamples were found, but these line graphs have rich and interesting endomorphisms and would surely repay further investigation!