

# Power graph, commuting graph, and all that

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# Introduction

There are many ways of building a graph from a group. For each of these, there are various natural questions, for example,

- ▶ To what extent does the graph determine the group?
- ▶ How do group-theoretic properties translate into graph-theoretic properties or *vice versa*?
- ▶ For which groups does the graph have some specified property?

I will be mainly concerned with the power graph, but I also discuss some variants of the power graph, the commuting graph, and the generation graph.

## Definitions

Let  $G$  be a group. In each of the following graphs, we take the vertex set of the graph to be the set  $G$ .

- ▶ In the **commuting graph**,  $x$  and  $y$  are joined if  $xy = yx$ .
- ▶ In the **power graph**,  $x$  and  $y$  are joined if one is a power of the other.
- ▶ In the **directed power graph**, there is an arc from  $x$  to  $y$  if  $y$  is a power of  $x$ .
- ▶ In the **enhanced power graph**,  $x$  and  $y$  are joined if there is an element  $z$  such that  $x$  and  $y$  are both powers of  $z$ .
- ▶ In the **generation graph**,  $x$  and  $y$  are joined if  $\langle x, y \rangle = G$ .

The definition of power graph is due to Kelarev and Quinn, while the enhanced power graph appears in my preprint with Aalipour, Akbari, Nikandish and Shaveisi (AACNS).

## Some properties

- ▶ The directed power graph is an orientation of the power graph (adding double edges between certain pairs of vertices if necessary).
- ▶ For the commuting and power graphs, if  $H$  is a subgroup of  $G$ , then the induced subgraph on  $H$  is the corresponding graph associated with  $H$ .
- ▶ This is slightly less obvious for the enhanced power graph, but can be seen by observing that  $x$  and  $y$  are joined in the enhanced power graph if and only if  $\langle x, y \rangle$  is cyclic.
- ▶ However, passing to a subgroup completely changes the generation graph: a proper subgroup of  $G$  is an independent set in the generation graph, even if it is a 2-generated group.
- ▶ The power graph is a (spanning) subgraph of the enhanced power graph, which is itself a subgraph of the commuting graph; and, in a non-abelian group, the generation graph is a subgraph of the *complement* of the commuting graph.

## Notation

From now on, I am mostly concerned with the power graph and its variants. We denote the power graph, directed power graph, and enhanced power graph by  $P(G)$ ,  $\vec{P}(G)$  and  $P_e(G)$ , and the commuting graph by  $\Gamma(G)$ .

## Determining the group

Non-isomorphic groups can have isomorphic power graphs. For finite groups, note that, if  $G$  is a group of exponent 3, then its power graph consists of a number of triangles sharing a common vertex; in the directed power graph, edges on the common vertex are directed towards it, while other edges are bidirectional. (Note that there do exist non-abelian groups of exponent 3.)

In the infinite case, things are even worse. For a prime  $p$ , let  $C_{p^\infty}$  be the group of all  $p$ -power roots of unity; alternatively it is the quotient of the additive group of rationals with  $p$ -power denominators by the subgroup of integers. Now clearly  $P(C_{p^\infty})$  is a countable complete graph; we cannot even determine the prime!

## Determining the directions

Again in the finite case, the power graph does not uniquely determine the directed power graph. Let  $G = C_6 = \langle a \rangle$ . Then the power graph consists of the complete graph  $K_6$  with the two edges  $\{a^2, a^3\}$  and  $\{a^4, a^5\}$  deleted. We cannot distinguish between the identity and the two generators  $a$  and  $a^5$  of the group.

### Theorem

*Let  $G$  be a finite group. Then  $P(G)$  determines  $\vec{P}(G)$  up to isomorphism.*

### Corollary

*Let  $G$  be a finite group. Then  $P(G)$  determines  $P_e(G)$  up to isomorphism.*

For  $x$  and  $y$  are joined in  $P_e(G)$  if and only if there is a vertex  $z$  which dominates both in  $\vec{P}(G)$ .

## For infinite groups

### Question

*Extend these results to some classes of infinite groups.*

In July, two undergraduates at St Andrews, Horacio Guerra and Šimon Jurina, showed that, for torsion-free abelian groups, the power graph determines the directed power graph up to isomorphism. They also made substantial progress on the problem of deciding when two such groups have isomorphic power graphs.



## Comparison

We have seen that the power graph is a subgraph of the enhanced power graph, which is itself a subgraph of the commuting graph. In AACNS, the following results appear.

### Theorem

*A finite group  $G$  has power graph equal to enhanced power graph if and only if every cyclic subgroup has prime power order. A group with this property is a  $p$ -group, a Frobenius group with kernel a  $p$ -group and complement a  $q$ -group, a 2-Frobenius group with normal subgroups  $F_1$  and  $F_2$  such that  $F_1$  and  $G/F_2$  are  $p$ -groups and  $F_2/F_1$  a  $q$ -group, or a group with a normal 2-subgroup with quotient group  $H$ , with  $S \leq H \leq \text{Aut}(S)$  and  $S \cong A_5$  or  $A_6$ .*

## Theorem

*A finite group has enhanced power graph equal to commuting graph if and only if its Sylow subgroups are cyclic or generalized quaternion. A group  $G$  with this property satisfies  $O(G)$  metacyclic,  $H = G/O(G)$  is a group with a unique involution  $z$ , and  $H/\langle z \rangle$  a cyclic or dihedral 2-group, a subgroup of  $P\Gamma L(2, q)$  containing  $PSL(2, q)$  for  $q$  an odd prime power, or  $A_7$ .*

## Corollary

*A finite group  $G$  with power graph equal to commuting graph is one of the following: a cyclic  $p$ -group; a semidirect product of  $C_{p^a}$  by  $C_{q^b}$ , where  $p$  and  $q$  are primes,  $a, b > 0$ , and  $C_{q^b}$  acts faithfully on  $C_{p^a}$ ; or a generalized quaternion group.*

The corollary was also extended to infinite soluble groups. No extension to arbitrary infinite groups seems likely, since for example any **Tarski monster** (an infinite group all of whose non-trivial proper subgroups are cyclic of a fixed prime order) satisfies the conditions.

## Question

*Can we extend the results of the first two theorems to infinite soluble groups?*

## Question

*What about graph-theoretic properties of the graphs  $\Gamma(G) \setminus P(G)$ ,  $P_e(G) \setminus P(G)$ ,  $\Gamma(G) \setminus P_e(G)$ ? In particular, when is one of these graphs connected (possibly after removing the identity)?*

## ... and the generation graph

For 2-generator non-abelian groups, as we saw, the generation graph is contained in the complement of the commuting graph.

### Question

*For which 2-generator non-abelian groups is the generation graph equal to the complement of the commuting graph?*

Such a group is generated by any pair of elements which don't commute. So every proper subgroup is abelian. Conversely, a non-abelian group with every proper subgroup abelian is generated by any two non-commuting elements, so the generation graph is the complement of the commuting graph. The finite groups with this property have been known since 1903 (Miller and Moreno). In the infinite case, the existence of Tarski monsters show things are more complicated!

## Question

*If  $G$  is 2-generator non-abelian but not a Miller–Moreno group, what can be said about the complement of the union of the commuting and generation graphs?*

It is now an easy exercise to answer the following:

## Question

*For which is a 2-generator non-abelian finite group is the complement of the generation graph equal to*

- ▶ *the enhanced power graph,*
- ▶ *the power graph?*

## Other issues

The first part of the following result is due to Shitov. A graph is **perfect** if every induced subgraph has clique number equal to chromatic number.

### Theorem

*The chromatic number of  $P(G)$  is at most countable. It is finite if and only if  $G$  has bounded exponent; in this case,  $P(G)$  is perfect.*

### Theorem

*A maximal clique in the enhanced power graph of a group is a subgroup, and is either cyclic or locally cyclic.*

## Some questions

The following open problems are taken from AACNS.

### Question

*For an infinite group  $G$ , is it true that the independence number of  $P(G)$  is finite if and only if  $G \cong C_{p^\infty} \times H$ , where  $p$  is prime and  $H$  a finite group?*

This is true for nilpotent groups.

### Question

*For which groups  $G$  are the induced subgraphs of the enhanced power graph and the commuting graph on  $G \setminus \{1\}$  equal?*

Note that free groups have this property.

# Removing dominating vertices

## Question

*Which groups have the property that the power graph is connected when the set of vertices dominating the graph is removed?*

In the case of the commuting graph, the dominating vertices are those in the centre, which are traditionally removed to investigate connectedness, as in recent results by Giudici, Morgan, Parker and others.

Let  $P^-(G)$  be the graph obtained by removing dominating vertices from  $P(G)$ .

## Proposition

*Let  $G$  be a cyclic group of order  $n$ . Then*

- ▶ *if  $n$  is a prime power, then  $P(G)$  is a complete graph, and  $G \setminus D(G) = \emptyset$ .*
- ▶ *if  $n$  is not a prime power, then  $P^-(G)$  is connected if and only if  $n$  is not the product of two primes.*



## Automorphism groups

The generation graphs of groups have huge automorphism groups. For example,  $\text{Aut}(\Gamma(A_5))$  has order 23482733690880. The reason is rather boring. If one generator has order 5, you can replace it by any non-identity power and generate the same group. Similarly for order 3. There are 6 subgroups of order 5 and 10 of order 3. So there is a normal subgroup of order  $(4!)^6(2!)^{10}$  fixing these classes. The quotient by this normal subgroup is  $S_5$ , which is the automorphism group of  $A_5$ . In a paper with Andrea Lucchini and Colva Roney-Dougal which just went on the arXiv, we conjecture that this holds more generally.

## A conjecture

Let  $G$  be a 2-generator group. Define an equivalence relation on  $G$ , where  $g \equiv h$  if  $g$  and  $h$  have the same neighbours in the generation graph. Let  $\bar{\Gamma}(G)$  be the vertex-weighted graph whose vertices are the equivalence classes, the weight being the size of the class, and adjacency as in  $\Gamma(G)$ .

### Question

*Is it true that, if  $G$  is a group in which every non-identity element lies in some 2-element generating set, we have*

$$\text{Aut}(\bar{\Gamma}(G)) = \text{Aut}(G)?$$

## Question

*What can be said about the automorphism group of the power graph, directed power graph, enhanced power graph, or commuting graph of a group  $G$ ?*

Note that these groups always contain the automorphism group of  $G$ : the main question is, how much bigger are they?

## Some papers

- ▶ Ghodratollah Aalipour, Saieed Akbari, Peter J. Cameron, Reza Nikandish and Farzad Shaveisi, On the structure of the power graph and the enhanced power graph of a group, arXiv 1603.04337.
- ▶ Peter J. Cameron, The power graph of a finite group II, *J. Group Theory* **13** (2010), 779–783.
- ▶ Peter J. Cameron, Andrea Lucchini and Colva M. Roney-Dougal, Generating sets of finite groups, arXiv 1609.06077.
- ▶ A. V. Kelarev and S. J. Quinn, A combinatorial property and power graphs of groups, *Contrib. General Algebra* **12** (2000) 229–235.
- ▶ Yaroslav Shitov, Coloring the power graph of a semigroup, arXiv 1607.00420.