Combinatorics of transformation semigroups and synchronization

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New Directions in Combinatorics
Institute for Mathematical Sciences
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May 2016

The dungeon

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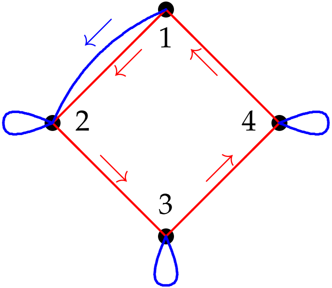

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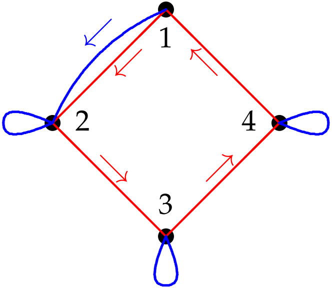

You can check that the sequence BRRRBRRRB will bring you to room 2 from any starting point.

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Algebraically, a transition is a transformation on the set of states; since we may compose transitions, an automaton is a transformation monoid on the set of states, with a prescribed set of generators.

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We can test whether an automaton is synchronizing in polynomial time, but finding the shortest reset word is NP-hard.

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One way round is clear; the other way, given a monoid M, define a graph where $v \sim w$ if and only if no element of M maps v and w to the same place, and check that this graph has the required property.

Synchronizing groups

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Our question now is:

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Which permutation groups are synchronizing?

This turns out to include many problems of great interest from extremal combinatorics and finite geometry, as I shall show you.

Synchronizing groups, 2

From our earlier theorem, we see:

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(The "trivial" graphs excluded are complete and null graphs.)

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For example, if G preserves a partition with m parts each of size k, then it preserves a complete multipartite graph, which has clique number equal to chromatic number.

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So, for a given group G, we may consider G-invariant graphs and their complements in complementary pairs: if a graph fails this test then it and its complement do not need to be considered further.
This also shows that a transitive permutation group of prime degree is synchronizing.

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If G has r orbits on unordered pairs of points of Ω, then there are $2^r - 2$ non-trivial G-invariant graphs to check. (Each orbit may consist of edges or non-edges; and two trivial graphs must be excluded.)

S_n acting on k-subsets, $k = 2$

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The first graph is the line graph of K_n: its clique number is $n - 1$, a maximal clique being all edges through a point. The second graph has clique number $\left\lfloor \frac{n}{2} \right\rfloor$.

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So S_n on 2-sets is synchronizing if and only if n is odd.

S_n acting on k-subsets, $k = 3$

The analogous result for S_n on the set of 3-subsets (for $n \geq 7$) is that it is synchronizing if and only if n is congruent to 2, 4 or 5 (mod 6) and $n > 8$. There are six invariant graphs; I will sketch the argument in two cases.

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Consider the graph whose vertices are the 3-sets, two 3-sets joined if and only if they have non-empty intersection. The clique number is $\binom{n-1}{2}$, a maximum clique consisting of all the 3-sets through a point. The chromatic number is $\binom{n-1}{2}$ if n is divisible by 3 (take a Baranyai partition of the 3-sets), and greater otherwise.

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Consider the graph whose vertices are the 3-sets, joined if and only if they intersect in at least two points. The clique number is $n - 2$, a maximum clique consisting of all 3-sets containing two given points. The chromatic number is at least $n - 2$; equality holds if and only if there is a large set of Steiner triple systems, a partition of the 3-sets into Steiner triple systems. These exist for all $n \equiv 1$ or 3 (mod 6) except for $n = 7$.

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It is clear from what is said above that it is going to involve ingredients like the Erdős–Ko–Rado theorem, Baranyai's theorem, Lovász's theorem on the chromatic number of Kneser graphs, the existence of t-designs and of large sets of t-designs for various parameters. Other considerations arise as well.
There is some beautiful combinatorics here, and I recommend the problem to anyone interested in a challenge!

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Hopefully it is clear from my remarks that testing synchronization for this class of groups will involve re-doing a lot of the combinatorics of sets and subspaces (including all the theorems mentioned earlier), in the setting of vector spaces and subspaces. Among this combinatorics, the existence of designs and large sets of designs will certainly feature prominently.

Classical groups on polar spaces

Another important class of groups consists of the classical groups (the symplectic, orthogonal and unitary groups), acting on the points of their associated polar spaces. The geometry associated with such a group is a non-degenerate bilinear, Hermitian or quadratic form on a vector space over a finite field: the points and lines of the polar space are the 1-dimensional and 2-dimensional subspaces on which the form is identically zero. So two points are collinear if and only if they are orthogonal with respect to the form.

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The automorphism groups are primitive and (except for the 4-dimensional orthogonal groups of split type acting on ruled quadrics) are also basic.

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A classical group acting on its polar space is non-synchronizing if and only if there exist either

- *an ovoid and a spread; or*
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Despite a lot of attention from finite geometers, we still do not know which polar spaces possess ovoids and/or spreads. The study of partitions into ovoids is more recent, partly motivated by this application to synchronization.

Almost synchronizing groups

In the examples so far, the maps not synchronized by primitive groups are uniform: all non-empty inverse images of points of the domain have the same size. People wondered if this were necessarily the case, and a permutation group G was said to be almost synchronizing if it synchronizes all non-uniform maps. It was conjectured that primitive groups are almost synchronizing.

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It was conjectured that primitive groups are almost synchronizing.
The conjecture is false, however. Recently we found counterexamples with a nice geometric structure. They are the automorphism groups of the Tutte–Coxeter and Biggs–Smith graphs, acting on the edge sets of these graphs. (Their line graphs have many non-uniform endomorphisms, and their endomorphism monoids have a rich structure worth investigating.)

Other connections with semigroups

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It is easy to see that idempotent generation of $\langle G, a\rangle$ for all rank 2 maps a implies that G is primitive on Ω; so we have a question about primitive permutation groups.

Road closures

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Let G be a primitive permutation group on Ω. Then $\langle G, a\rangle$ is idempotent-generated for all rank 2 maps a if and only if G has the following property: for any orbital graph of G, with edge set E, and any block of imprimitivity B for the action of G on E, the graph with edge set $E \setminus B$ is connected.

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In other words, thinking of the orbital graph as a road network, we cannot disconnect it by closing the roads in some block of imprimitivity for G.

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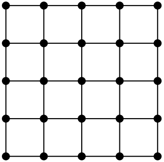

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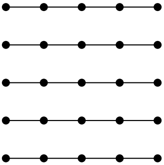

If workmen come and dig up all the vertical roads, then it is impossible to get from one row to another.

A conjecture

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Let G be a basic primitive permutation group. Suppose that G does not have an imprimitive normal subgroup of index 2, and is not one of the triality examples just mentioned. Then G has the 2-Hc property. Hence, for any rank 2 map a, the semigroup $\langle G, a \rangle \setminus G$ is idempotent-generated.

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There are two kinds of basic examples for which the road closure condition fails. The first are a rich and varied class consisting of primitive groups which have imprimitive normal subgroups of index 2. The other form a very limited class based on the geometric phenomenon of triality.

Conjecture

Let G be a basic primitive permutation group. Suppose that G does not have an imprimitive normal subgroup of index 2, and is not one of the triality examples just mentioned. Then G has the 2-Hc property. Hence, for any rank 2 map a, the semigroup $\langle G, a \rangle \setminus G$ is idempotent-generated.

This conjecture has been checked computationally for all degrees up to 130 and many larger degrees. No counterexamples have been found.