The random graph and its friends

Peter J. Cameron
University of St Andrews

Mathematics Colloquium
University of Vienna
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Example: the Petersen graph

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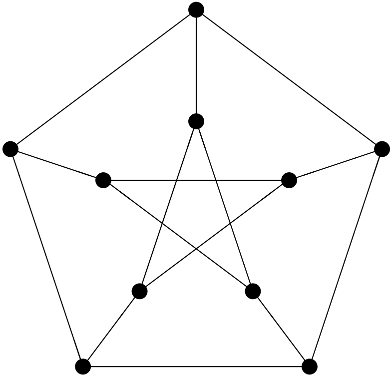

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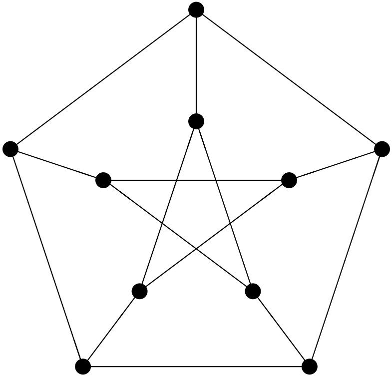

I am going to tell you about the most famous *infinite* graph …

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I will tell you some of its story.

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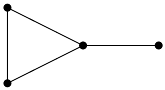

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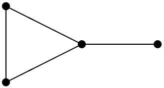

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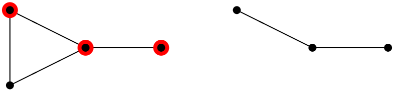

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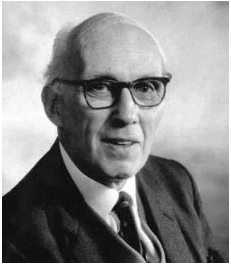
Rado's universal graph



In 1964, Richard Rado published a construction of a countable graph which was universal. This means that every finite or countable graph occurs as an induced subgraph of Rado's graph.

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The vertex set of Rado's graph R is the set $\mathbb{N}$ of natural numbers (including 0).

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Theorem

There is a countable graph R with the following property: if a random graph X on a fixed countable vertex set is chosen by selecting edges independently at random with probability $\frac{1}{2}$, then the probability that X is isomorphic to R is equal to 1.

The proof

I will show you the proof.

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I claim that one of the distinguishing features of mathematics is that you can be convinced of such an outrageous claim by some simple reasoning. I do not believe this could happen in any other subject.

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The proof depends on the following property, which a graph may or may not possess:

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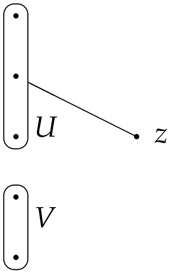

The point z is called a witness for the sets U and V.

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Then you will be convinced!

Proof of Fact 1

We use from measure theory the fact that a countable union of null sets is null. We are trying to show that a countable graph fails $(*)$ with probability 0; since there are only countably many choices for the (finite disjoint) sets U and V, it suffices to show that for a fixed choice of U and V the probability that no witness z exists is 0.

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Since all edges are independent, the probability that none of $z_1, z_2, \ldots, z_N$ is the required witness is $\left(1 - \frac{1}{2^n}\right)^N$, which tends to 0 as $N \to \infty$.

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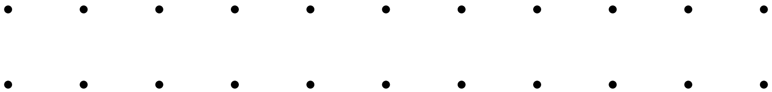
So the event that no witness exists has probability 0, as required.

Proof of Fact 2

We use a method known to logicians as "back and forth". Suppose that Γ_1 and Γ_2 are countable graphs satisfying $(*)$: enumerate their vertex sets as $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$. We build an isomorphism ϕ between them in stages.



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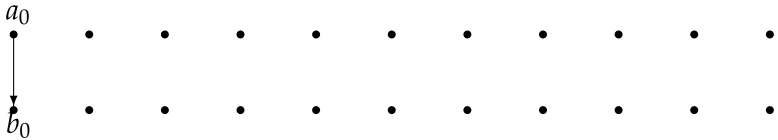

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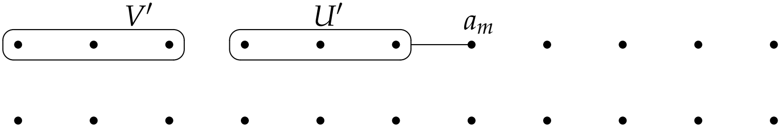

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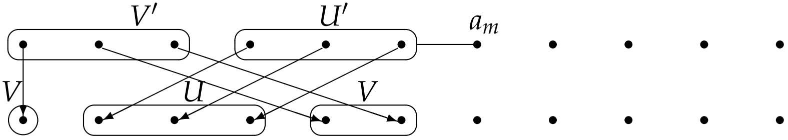

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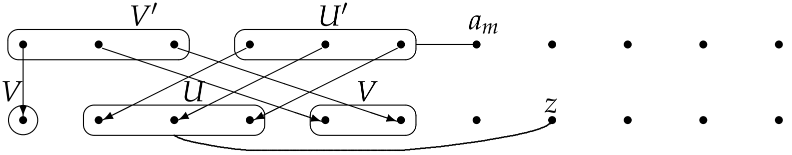

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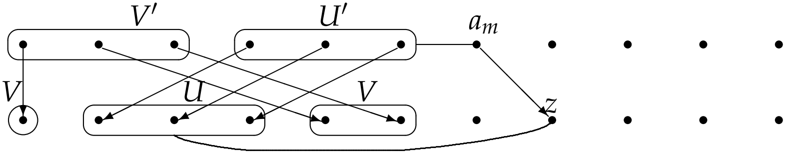

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The back-and-forth method is often credited to Georg Cantor, but it seems that he never used it, and it was invented later by E. V. Huntington.

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To see this, take $\Gamma_1 = \Gamma_2 = R$, and start the back-and-forth machine from the given finite isomorphism.

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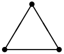
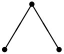
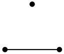
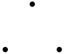
Graph				
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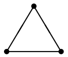
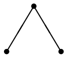
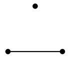
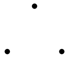
Graph				
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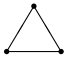
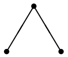
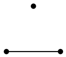
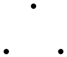
Graph				
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In fact, the probability of any non-trivial symmetry tends rapidly to 0 as the number of vertices increases.
Indeed, the theorem of Erdős and Rényi was a short appendix to a long paper showing that most *finite* graphs are "as far from symmetry" as possible.

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A number-theoretic construction

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Recall that, if p is an odd prime not dividing a, then a is a quadratic residue (mod p) if the congruence $a \equiv x^2 \pmod{p}$ has a solution, and a quadratic non-residue otherwise. A special case of the law of quadratic reciprocity, due to Gauss, asserts that if the primes p and q are congruent to 1 (mod 4), then p is a quadratic residue (mod q) if and only if q is a quadratic residue (mod p).

A number-theoretic construction

Since the prime numbers are "random", we should be able to use them to construct the random graph. Here's how.

Recall that, if p is an odd prime not dividing a, then a is a quadratic residue (mod p) if the congruence $a \equiv x^2 \pmod{p}$ has a solution, and a quadratic non-residue otherwise. A special case of the law of quadratic reciprocity, due to Gauss, asserts that if the primes p and q are congruent to 1 (mod 4), then p is a quadratic residue (mod q) if and only if q is a quadratic residue (mod p).

So we can construct a graph whose vertices are all the prime numbers congruent to 1 (mod 4), with p and q joined if and only if p is a quadratic residue (mod q): the law of quadratic reciprocity guarantees that the edges are undirected.

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To show this we have to verify condition $(*)$. So let U and V be finite disjoint sets of primes congruent to 1 (mod 4). For each $u_i \in U$ let a_i be a fixed quadratic residue (mod u_i); for each $v_j \in V$, let b_j be a fixed quadratic non-residue mod v_j.

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By the Chinese Remainder Theorem, the simultaneous congruences

- $z \equiv a_i \pmod{u_i}$ for all $u_i \in U$,
- $z \equiv b_j \pmod{v_j}$ for all $v_j \in V$,
- $z \equiv 1 \pmod 4$,

have a solution modulo $4 \prod u_i \prod v_j$. By Dirichlet's Theorem, this congruence class contains a prime, which is the required witness.

The Skolem paradox

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The Skolem paradox is this: There is a theorem of set theory (for example, as axiomatised by the Zermelo–Fraenkel axioms) which asserts the existence of uncountable sets. Assuming that ZF is consistent (as we all believe!), how can this theory have a countable model?

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My point here is not to resolve this paradox, but to use it constructively.

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Let M be a countable model of the Zermelo–Fraenkel axioms for set theory. Then M consists of a collection of things called "sets", with a single binary relation $\in$, the "membership relation".

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Indeed, the precise form of the axioms is not so important. We need a few basic axioms (Empty Set, Pairing, Union) and, crucially, the Axiom of Foundation, and that is all. It does not matter, for example, whether or not the Axiom of Choice holds.

Back to Rado's graph

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There is a simple description of a model of set theory in which the negation of the axiom of infinity holds (called hereditarily finite set theory). We represent sets by natural numbers. We encode a finite set $\{a_1, \ldots, a_r\}$ of natural numbers by the natural number $2^{a_1} + \cdots + 2^{a_r}$. (So, for example, 0 encodes the empty set.)

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When we apply the construction of "symmetrising the membership relation" to this model, we obtain Rado's description of his graph!

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In fact, fifteen years earlier, Roland Fraïssé had asked the question: which homogeneous relational structures exist? Fraïssé defined the age of a relational structure M to be the class $\mathrm{Age}(M)$ of all finite structures embeddable in M (as induced substructure). In terms of this notion, he gave a necessary and sufficient condition.

Fraïssé classes and Fraïssé limits

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A class $\mathcal{C}$ satisfying these conditions is a Fraïssé class, and the countable homogeneous structure M is its Fraïssé limit.

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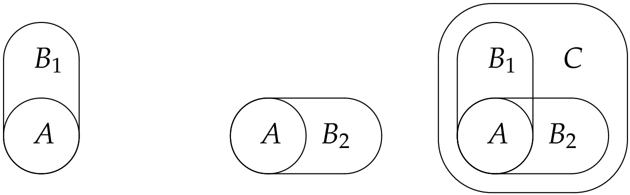

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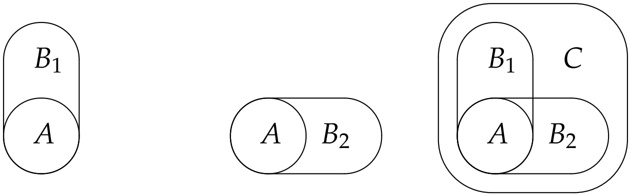

Note that the intersection of B_1 and B_2 may be larger than A.

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Each of the following classes is a Fraïssé class; the proofs are exercises. Thus the corresponding universal homogeneous Fraïssé limits exist.

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There are *many* others!

Rolling back further

A quarter of a century earlier, these ideas had already been used by the Soviet mathematician P. S. Urysohn. He visited western Europe with Aleksandrov, talked to Hilbert, Hausdorff and Brouwer, and was drowned while swimming in the sea at Batz-sur-Mer in south-west France at the age of 26 in 1924.

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Urysohn's theorem

A Polish space is a metric space which is complete (Cauchy sequences converge) and separable (there is a countable dense set). A metric space M is homogeneous if any isometry between finite subspaces extends to an isometry of M.

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This unique metric space is known as the Urysohn space. Its study has been popularised in recent years by Anatoly Vershik.

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Its completion is easily seen to be the required Polish space.

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- The class of integral metric spaces, those with all distances integers. The Fraïssé limit is a kind of universal distance-transitive graph.
- The class of metric spaces with all distances 1 or 2. The Fraïssé limit is the random graph!

Let M be the Fraïssé limit of the class of metric spaces with all distances 1 or 2; form a graph by joining two points if their distance is 1. Since the graph is homogeneous, if v and w are two vertices at distance 2, there is a vertex at distance 1 from both. Thus the distance in M coincides with the graph distance in this graph. The graph is universal and homogeneous, and so is R.

Reversing the arrows

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A category theorist, looking at Fraïssé's construction, would draw a commutative diagram, with arrows representing embeddings.
She would then ask for a similar theorem, with the arrows reversed, and representing projections.
There is such a theorem, the projective Fraïssé theorem. Rather than describe it in detail, I will give you one application.

Cantor space

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Topologically, it is homeomorphic to a product of countably many copies of a 2-element discrete space, and hence it is compact and totally disconnected.
If we replace 0 and 2 in base 3 by 0 and 1 in base 2, we make a huge difference to the topology: we now get the unit interval, which is connected. This change occurs because a few real numbers have two base 2 representations, and so the map from Cantor space to the unit interval is not quite one-to-one.

The pseudo-arc

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We have to be careful about the word “typical”. In a probability space this can mean “a set of measure 1”, but here we don’t have a measure. Instead we use a notion from Baire category: in a complete metric space, a set is residual if it contains a countable intersection of open dense subsets. Residual sets behave like complements of null sets: they are non-empty, meet every open set, and any two (or countably many) of them intersect in a residual set.

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The metric we use on closed subsets of the square is Hausdorff metric: two sets are within distance ϵ if every point of one is within distance ϵ from some point of the other.

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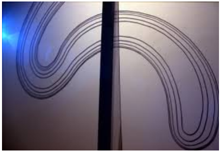

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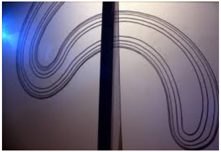

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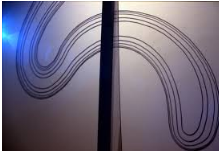

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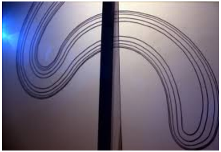

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Moreover, the statement in the first paragraph remains true if we replace the unit square by the unit hypercube in $\mathbb{R}^n$ for any $n \geq 2$, or in Hilbert space.
Its topological definition might suggest that it cannot be constructed by discrete methods, but this is not so …

. . . as projective Fraïssé limit (almost)

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Using this, Solecki and Tsankov were able to give a new proof of Bing's theorem that the pseudo-arc has a transitive homeomorphism group.