The ADE affair

Peter J. Cameron
University of St Andrews

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March 2017

Calling card

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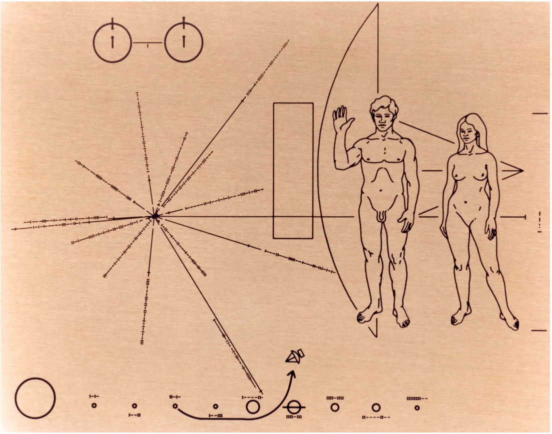
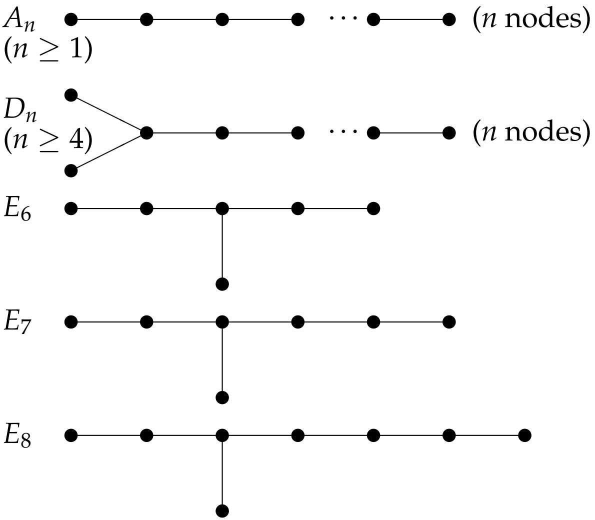
The Pioneer engineers decided that this would do:



Francis Buekenhout had another idea:



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- … the Coxeter groups generated by reflections, or of Weyl groups with roots of equal length.

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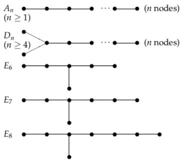
I will say a bit about all this.

The extended diagrams



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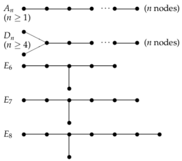

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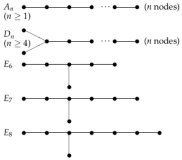

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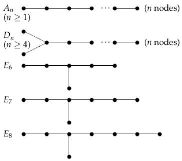

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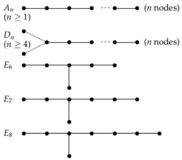
- for A_n, the new vertex is joined to both ends of the path, forming an $(n+1)$-cycle;
- for D_n, it creates a fork at the other end of the diagram, or a $K_{1,4}$ if $n = 4$;
- for E_n, it extends one of the arms, so that the numbers of vertices on the arms are $(3, 3, 3)$ $(n = 6)$, $(2, 4, 4)$ $(n = 7)$, or $(2, 3, 6)$ $(n = 8)$.

What are these diagrams?



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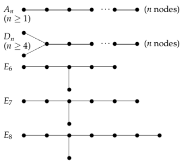

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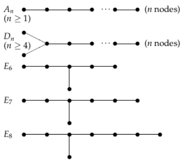

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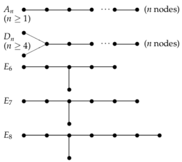

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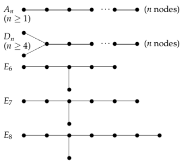

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Although this theorem was proved in 1969, it is in some sense implicit in the classification of simple Lie algebras over $\mathbb{C}$.

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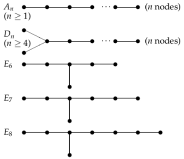
In particular, it does not contain $\tilde{A}_n$ (a cycle), and so is a tree; it does not contain $\tilde{D}_n$, and so has at most one branchpoint, such a point having valency at most 3; and it does not contain $\tilde{E}_n$, and so the lengths of the three arms are restricted to the appropriate values.

Polyhedra and tessellations



The numbers of vertices on the arms of the E_n diagrams are $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ for $n = 6, 7, 8$ respectively. This should remind you of the regular polyhedra in 3-space.

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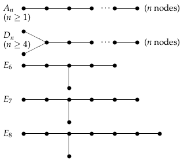

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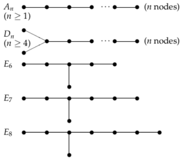

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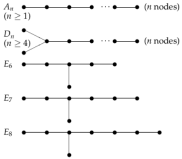
These are not accidental; we will return to them later.

A detour



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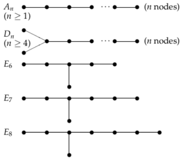

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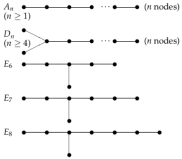

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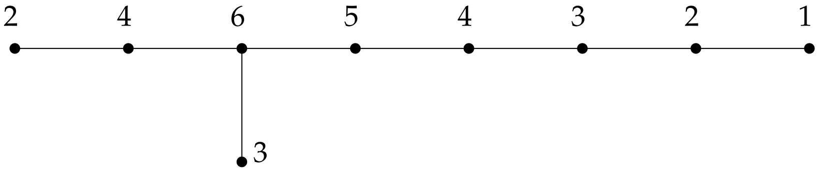
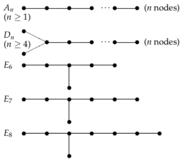
It is a simple exercise to write down such a labelling. See the next slide for $\tilde{E}_8$.

An eigenvector



Finite rotation groups



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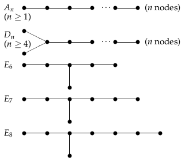

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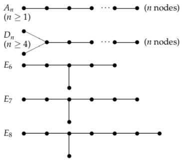

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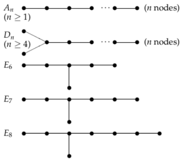

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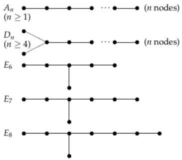

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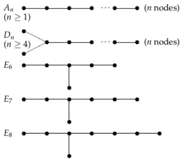

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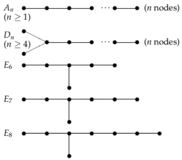

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The correspondence with the ADE diagrams is clear.

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Each of these groups comes with a "natural" two-dimensional unitary representation ρ, which is in fact self-dual (implying that $\rho \otimes \rho$ contains the trivial representation).

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We conclude that the graph is an extended ADE diagram.

Indeed, the correspondence agrees with the one we just observed for the rotation groups.

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I am afraid I am not competent to tell you about this, but I am very happy to have had a part in the story!

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On comparing their work, each of them had missed a different infinite family. So it was clearly an ADE problem, as I will now describe.

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- $\sigma_v(w) = w - 2(v \cdot w)/(v \cdot v)v$, an integer linear combination of v and w. From this, it can be shown that R spans a lattice (a root lattice).

Roots of constant length

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It can be shown that there is a basis consisting of roots with non-positive inner products. If G is the Gram matrix, then (after normalisation) we have $G = 2I - A$, where A is the adjacency matrix of a graph. Since G is positive definite, A has greatest eigenvalue less than 2.

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Thus, if the root system is indecomposable (that is, the graph is connected), it is an ADE diagram. Moreover, the graph determines the root system.

From the icosahedron to E_8

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Is there a direct connection between the 3-dimensional rotation groups and the root systems, underlying the McKay correspondence? A beautiful construction was found recently by Pierre Dechant.
This uses the *Clifford algebra*, a deformation of the exterior algebra: we consider only the case of real vector spaces V with a positive definite inner product. The multiplication in the Clifford algebra is given by

$$xy = x \cdot y + x \wedge y,$$

where the inner and outer (exterior) product are the symmetric and antisymmetric parts of the Clifford algebra product. The dimension of the Clifford algebra is the same as that of the exterior algebra on V, namely 2^n, where $n = \dim(V)$.

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The group of reflections and rotations of the icosahedron has order 120, and so it lifts to a set of 240 vectors in $\mathbb{R}^8$. These form the E_8 root system.
There is much more to the story, but no time to tell it here ...

The remaining root systems

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  - one root system has type D_n and the other is an orthonormal basis, giving types B_n or C_n (depending on which roots are longer); or

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- If the ratio of squared lengths is 2, then either
  - one root system has type D_n and the other is an orthonormal basis, giving types B_n or C_n (depending on which roots are longer); or
  - the two root systems have type D_4, and together give type F_4.

Lie algebras

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Then appeal to the classification of root systems, and show that the root system uniquely determines the algebra.
The dimension of the Lie algebra arising from the root system R is $\dim(R) + |R|$ (the first term for the Cartan algebra, and the second for the root subspaces). For E_8 we obtain $8 + 240 = 248$.

Star-closed sets

To classify the graphs with least eigenvalue -2, we reverse the procedure.

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Let A be the adjacency matrix of such a graph. Then $A + 2I$ is positive semidefinite, and so is the matrix of inner products of a set of vectors in $\mathbb{R}^n$, where n is the comultiplicity of -2 as an eigenvalue. Clearly any two of these vectors make an angle 90° or 60°.

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Now a geometric argument shows that a set of lines through the origin in $\mathbb{R}^n$, in which any two lines make angle 90° or 60°, and is maximal with respect to this property, is star-closed: that is, if two lines in the set make angle 60°, then the third line in their plane at 60° to both is also in the set.

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Taking vectors of fixed length on the lines in such a star-closed set, we obtain a root system!

The classification

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Theorem

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The paper containing this theorem (with Goethals, Seidel, and Ernest Shult) is one of my most cited.

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It follows from the commutation relations for such subgroups that, if two of them commute, then the corresponding roots make an angle at most 90°.
So subgroups with the required property are classified by the graphs with least eigenvalue -2, and the preceding theorem applies.

And more . . .

I have not told you the whole story. There are connections with algebras of finite representation type, with critical points of smooth functions, and with cluster algebras (which come up in the theories of Poisson algebras and totally positive matrices).

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I have not told you the whole story. There are connections with algebras of finite representation type, with critical points of smooth functions, and with cluster algebras (which come up in the theories of Poisson algebras and totally positive matrices). Indeed, when Fomin and Zelevinsky invented cluster algebras, they didn't at first know that the finite dimensional ones fitted the ADE classification, but discovered this later.

Cluster algebras: a taster

Consider the sequence given by the recurrence

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}.$$

It is easy to see that it returns to its initial value after five steps. If the first two terms are $1, 1$, then the sequence runs $1, 1, 2, 3, 2, 1, 1, \ldots$.

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It is easy to see that it returns to its initial value after five steps. If the first two terms are $1, 1$, then the sequence runs $1, 1, 2, 3, 2, 1, 1, \ldots$.
However the related recurrence

$$x_{n+3} = \frac{1 + x_{n+1}x_{n+2}}{x_n}$$

starting $1, 1, 1$, runs $1, 1, 1, 2, 3, 7, 11, 26, 41, 97, 153, 362, 571, \ldots$, and grows forever. This sequence has many interesting interpretations, including the denominators of the continued fraction convergents to $\sqrt{3}$.

Each of these examples is associated with a cluster algebra associated with a quiver (oriented graph). In the first case it is an orientation of A_2 (a single edge); the second case is an oriented triangle, so not an ADE diagram. This agrees with the finiteness theorem of Fomin and Zelevinsky.

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Note too that, despite the denominators in the recurrence, the sequences generated consist of integers. This is explained by the theory.
But note that the period 5 in the first case seems to have nothing to do with the root system, or reflection group, or anything else, traditionally associated with A_2. There are groups here too, which are not the Coxeter groups of the appropriate types. For the second example, we get the group of rotations of the icosahedron …

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