Permutation Groups and Transformation Semigroups
Lecture 3: Regularity

Peter J. Cameron
University of St Andrews

Shanghai Jiao Tong University
November 2017

The prototype result

Recall the theoreom of Araújo, Mitchell and Schneider in Lecture 1. A semigroup is regular of each of its elements has a (von Neumann) inverse.

Theorem

Let G be a permutation group on Ω, with $|\Omega| = n$. Suppose that, for any map f on Ω which is not a permutation, the semigroup $\langle G, f \rangle$ is regular. Then either G is the symmetric or alternating group on Ω, or one of the following occurs:

- $n = 5$, $G = C_5$, $C_5 \rtimes C_2$, *or* $C_5 \rtimes C_4$;
- $n = 6$, $G = \mathrm{PSL}(2,5)$ *or* $\mathrm{PGL}(2,5)$;
- $n = 7$, $G = \mathrm{AGL}(1,7)$;
- $n = 8$, $G = \mathrm{PGL}(2,7)$;
- $n = 9$, $G = \mathrm{PGL}(2,8)$ *or* $\mathrm{P\Gamma L}(2,8)$.

The problem

Our goal is to strengthen this theorem, by requiring regularity of $\langle G, f \rangle$ only for some choices of f. Targets in increasing order of difficulty could include:

- all maps f with $\mathrm{rank}(f) = k$, for some k;
- all maps f with $\mathrm{Im}(f) = A$, for some fixed k-set A;
- a single map f.

We are a long way from a definitive result on the last case, but there has been substantial progress on the other two. This is today's topic.

Transitivity and homogeneity

As we saw, (CFSG) has the consequence that, for $k \geq 2$, the finite k-transitive groups are all known explicitly. The lists (other than symmetric and alternating groups) are finite for $k = 4, 5$ and infinite for $k = 2, 3$.
Much earlier, Livingstone and Wagner had investigated the relationship between k-homogeneity and k-transitivity. (We remarked earlier that k-transitivity implies k-homogeneity.) A group of degree n is k-homogeneous if and only if it is $(n-k)$-homogeneous; so we may assume that $k \leq n/2$. Now Livingstone and Wagner proved the following theorem by elementary methods:

The Livingstone–Wagner theorem

Theorem

Suppose that $k \leq n/2$, and let G be k-homogeneous of degree n. Then

- *G is $(k-1)$-homogeneous;*
- *G is $(k-1)$-transitive;*
- *if $k \geq 5$, then G is k-transitive.*

Subsequently, Kantor determined all the k-homogeneous but not k-transitive groups for $k = 2, 3, 4$. (There are infinitely many for $k = 2, 3$ but only finitely many for $k = 4$.) These arguments do not use CFSG.

k-homogeneous implies $(k-1)$-homogeneous

The function on a permutation group which maps an element g to its number $\mathrm{fix}(g)$ of fixed points is a character of G, the trace of a matrix representation.

The character theory of finite symmetric groups is a classical subject. In particular, there are irreducible characters χ_i for $0 \le k \le n/2$ such that the permutation character π_k of the action on k-sets is given by

$$\pi_k = \sum_{i=0}^{k} \chi_i.$$

In particular, π_{k-1} is a constituent of π_k.

Now restrict to a group G. If G is k-homogeneous, it is transitive on k-sets, and so $\pi_k|_G$ contains the trivial character with multiplicity 1. Since π_{k-1} is a constituent, the multiplicity of the trivial character in it must also be 1, whence G is transitive on $(k-1)$-sets, that is, $(k-1)$-homogeneous.

The k-homogeneous, not k-transitive groups

The case $k = 2$: If G is 2-homogeneous but not 2-transitive, then G has odd order (because if $|G|$ is even then some pair, and hence every pair, would be interchanged by an involution in G). Hence G is solvable (by the Feit–Thompson Theorem, and so is an affine group: its minimal normal subgroup is the group of translations of a finite vector space.
Then $\langle G, -I \rangle$ is 2-transitive and has G as a subgroup of index 2. Using the earlier classification of solvable 2-transitive groups leads to the identification of G.
The case $k = 3$: This falls into two types: one consists of groups having a normal subgroup $\mathrm{PSL}(2, q)$ for some odd q (a transitive extension of the previous case); the other has just three groups, of degrees 8, 8 and 32.
The case $k = 4$: We only get transitive extensions of the second case above, with degrees 9, 9 and 33.

Regularity

We need to understand what it means that the map f is regular in $\langle G, f \rangle$.
Suppose that f is regular, with von Neumann inverse h. Say

$$h = g_1 f g_2 \cdots g_{m-1} f g_m,$$

so that

$$f g_1 f g_2 \cdots g_{m-1} f g_m f = f.$$

Hence

$$\mathrm{rank}(f) \geq \mathrm{rank}(f g_1 f) \geq \mathrm{rank}(f g_1 f g_2 \cdots g_{m-1} f g_m) = \mathrm{rank}(f),$$

so equality holds throughout.
Now from $\mathrm{rank}(f) = \mathrm{rank}(f g_1 f)$ it follows that g_1 maps $\mathrm{Im}(f)$ to a transversal for $\mathrm{Ker}(f)$.

Conversely suppose that g is a permutation such that $\mathrm{Im}(fg)$ is a transversal to $\mathrm{Ker}(f)$. Then fg induces a permutation of $\mathrm{Im}(f)$, and some power of it is the identity on $\mathrm{Im}(f)$, so that $(fg)^m$ is an idempotent. Then

$$(fg)^m f = f,$$

so f is regular in $\langle f, g \rangle$, with von Neumann inverse $g(fg)^{m-1}$. We conclude:

Theorem

The map f is regular in $\langle f, G \rangle$ if and only if G contains an element which maps $\mathrm{Im}(f)$ to a transversal for $\mathrm{Ker}(f)$.

The universal transversal property

Let k be a positive integer less than n, and G a permutation group of degree n. We say that the permutation group G has the k-universal transversal property if, for any k-set A and any k-partition P, there is an element $g \in G$ such that Ag is a transversal for P.

It follows from the preceding result that:

Theorem

For the permutation group G and positive integer k, the following are equivalent:

- *for every map f of rank k, f is regular in $\langle G, f\rangle$;*
- *G has the k-universal transversal property.*

This is because, given any k-set A and k-partition P, there is a map of rank k whose image is A and whose kernel is P.

So our first target requires the classification of groups with the k-universal transversal property (or k-ut property, for short).

A surprise

Araújo and I proved the following theorem:

Theorem (CFSG)

Given k with $1 \leq k \leq n/2$, the following conditions are equivalent for a subgroup G of S_n:

- *For any rank k map f, f is regular in $\langle G, f \rangle$.*
- *For any rank k map f, $\langle G, f \rangle$ is regular (this means that all its elements are regular).*
- *For any rank k map f, f is regular in $\langle g^{-1}fg : g \in G \rangle$.*
- *For any rank k map f, $\langle g^{-1}fg : g \in G \rangle$ is regular.*
- *G has the k-universal transversal property.*

As we saw, the first and fifth conditions are equivalent. Also the equivalence of the first and third, and of the second and fourth, was already known to semigroup theorists.
But why does regularity for elements of rank k imply regularity for elements of smaller rank?

k-ut implies $(k-1)$-ut

The heart of our proof is:

Theorem (CFSG)

For $2 \leq k \leq n/2$, the k-ut property implies the $(k-1)$-ut property.

This is reminiscent of the first part of the Livingstone–Wagner theorem. We spent some time looking for an elementary combinatorial proof of this fact, but didn't succeed.

Instead, our argument for this simple fact comes close to a complete classification of the permutation groups with the k-ut property. So, indeed, its proof depends on CFSG.

Primitivity revisited

Theorem

For a permutation group G on Ω, with $|\Omega| = n > 2$, the following conditions are equivalent:

- *G is primitive (that is, G preserves no non-trivial partition of Ω);*
- *every non-trivial G-invariant graph on Ω is connected;*
- *G has the 2-ut property;*
- *for any map f of rank $n - 1$, $\langle G, f \rangle$ is synchronizing.*

The equivalence of 2-ut and primitivity show that we cannot hope for a complete classification of 2-ut groups. We'll look at k-ut for $k > 2$.

Proofs

Proof.
The connected components of a disconnected G-invariant graph form a G-invariant partition. Conversely, if G preserves a non-trivial partition, then it preserves the disjoint union of complete graphs on its parts. So the first two conditions are equivalent.
The 2-ut property says that an orbit of G on 2-sets (the edge set of a minimal non-null G-invariant graph) contains a transversal to any 2-partition; so any such graph is connected. So the second and third conditions are equivalent.
The equivalence of the last condition is a theorem of Rystsov; the proof is an exercise. (This gives another proof that synchronizing groups are primitive.) □

(l,k)-homogeneity

Our argument uses a generalisation of k-homogeneity. We say that the permutation group G on Ω is (l,k)-homogeneous for $l \leq k$ if given any k-subset K and l-subset L of Ω, there exists $g \in G$ such that $Lg \subseteq K$.

- (k,k)-homogeneity is equivalent to k-homogeneity.
- A group with k-ut is $(k-1,k)$-homogeneous. [For if $|L| = k-1$, consider the partition P into the singletons of L and all the rest. Then G contains an element g such that Kg is a transversal to P; so $Lg^{-1} \subseteq K$.]
- A $(k-1)$-homogeneous group is clearly $(k-1,k)$-homogeneous. Conversely, we will now see that a $(k-1,k)$-homogeneous group of sufficiently large degree is $(k-1)$-homogeneous.

The Ramsey argument

The Ramsey number $R_{k-1}(k,k)$ is the smallest number n such that, if the $(k-1)$-subsets of an n-set are coloured red and blue in any manner, then there is a monochromatic set of size k (one such that all its $(k-1)$-subsets have the same colour).
Let $f(k)$ be the Ramsey number $R_{k-1}(k,k)$. Suppose that $n \geq f(k)$, and let G be a permutation group of degree n which is $(k-1,k)$-homogeneous. Suppose that G is not $(k-1)$-homogeneous. Then it has at least two orbits on the set of $(k-1)$-subsets; choose one orbit and colour its members red, and colour the other $(k-1)$-sets blue.
Since $n \geq R_{k-1}(k,k)$, there exists a monochromatic k-set. But since G is $(k-1,k)$-homogeneous, every k-set contains representatives of all the G-orbits on $(k-1)$-sets, and in particular contains both red and blue sets.
This is a contradiction!

The k-ut property

Pushing harder, we show that a $(k-1, k)$-homogeneous group is $(k-1)$-homogeneous with a few small exceptions. For $k > 2$, as explained, we know (from CFSG and Kantor's work) the list of $(k-1)$-homogeneous groups. The result is an almost complete classification of groups with k-ut for $2 < k \leq n/2$. A few stubborn families, including the Suzuki groups $\mathrm{Sz}(q)$ for $k = 3$, are still holding out, and provide an interesting challenge for group theorists.

The k-existential transversal property

In order to study groups G such that $\langle G, f \rangle$ is regular for all maps f with a prescribed image (instead of a prescribed rank), we need a weakening of k-ut, defined as follows.

The permutation group G on Ω has the k-existential transversal property, or k-et for short, if there exists a set A of size k such that, for any k-partition P, some image of A is a transversal for P. The set A is called a witnessing set for G.

This problem poses further group-theoretic and combinatorial challenges, which I will describe briefly.

General results

Groups with k-et need not be transitive; but, for $2 < k < n$, an intransitive group with k-et fixes a point and acts $(k-1)$-homogeneously on the remaining points. So these groups are known.
Transitive groups with k-et for $4 \leq k \leq n/2$ must be primitive with finitely many exceptions.
However, there is a shock in store. It is not true that k-et implies $(k-1)$-et. In fact just two counterexamples are known: these are the 3-transitive groups $\mathrm{AGL}(4,2) = 2^4 \rtimes A_8$ and its subgroup $2^4 \rtimes A_7$. These groups have 4-et and 6-et but not 5-et. It is conjectured that they are the only exceptions. Indeed, we hope to classify k-et groups for $k \geq 4$.

Tools

Suppose that G is k-et, with witnessing set A. Then every G-orbit on $(k-1)$-sets has a representative in A (take partitions consisting of $k-1$ singletons and the rest in a single part). So there are at most k such orbits, and

$$|G| \geq \binom{n}{k-1} \Big/ k.$$

In fact this bound can be improved, roughly by a factor of 2. However, one of the striking consequences of CFSG is that *primitive groups are small*, or order at most $n^{1+\log n}$ with "known" exceptions. Confronting these two bounds gives strong restrictions.
Combinatorial tools such as Ramsey's theorem are also useful.

Results

Theorem (CFSG)

A transitive group with the k-et property, for $8 \leq k \leq n/2$, is symmetric or alternating.

This is best possible, since the Mathieu group M_{24} has 7-et but not 8-et. However, we expect that it will be the only one.

In order to go from "f is regular in $\langle G, f \rangle$" to "$\langle G, f \rangle$ is regular", we would like to know that k-et implies $(k-1)$-ut with known exceptions. We would then need to be able to handle these exceptions, using information about the group to restrict the maps f that need to be considered.

Further work

We hope to complete the classification of groups with the k-et property for $k \geq 4$. The case $k = 3$ may be more challenging, but $k = 2$ is hopeless.
Recall that the 2-ut property is equivalent to primitivity. The 2-et property is weaker. Let us say that a group G is totally imprimitive if any pair of distinct points a and b of Ω are contained together in a proper block of imprimitivity for G.

Proposition

The permutation group G has the 2-et property if and only if it is not totally imprimitive.

Another thing we would very much like to do is to find an “elementary” proof (not using CFSG) that the k-ut property implies the $(k-1)$-ut property for $k \leq n/2$.