Permutation Groups and Transformation Semigroups
Lecture 4: Idempotent generation

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Idempotent generation

We are interested in when a semigroup S is idempotent-generated, that is, generated by the idempotents it contains.
Of course, a group contains only one idempotent; so a semigroup whose units form a non-trivial group cannot be idempotent generated. Thus we can pose our problem as follows:
For which groups $G \leq \mathrm{Sym}(\Omega)$ is it true, that for all (or all satisfying some restriction) maps f, the semigroup $\langle G, f\rangle \setminus G$ is idempotent-generated?
Requiring this for all $f \in T(\Omega) \setminus \mathrm{Sym}(\Omega)$, the question was settled by Araújo, Mitchell and Schneider in the paper mentioned earlier.
So we will typically require this to hold for all maps of some given rank k.

Idempotent generation and k-ut

We saw in the last lecture that the following are equivalent:

- G has the k-ut property;
- for any map f of rank k, $\langle G, f \rangle \setminus G$ contains an idempotent of rank k.

So the condition "for all maps f of rank k, $\langle G, f \rangle \setminus G$ is idempotent-generated" is a strengthening of k-ut. (We will call this the k-ig property.)

In general, we don't have a precise equivalent of k-ig. We defined a notion of strong k-ut which implies k-ig, and obtained some results this way.

The case $k = 2$

The most interesting case is the case $k = 2$.
We saw in the last lecture that 2-ut is equivalent to primitivity, which is also equivalent to the connectedness of all non-empty G-invariant graphs.
So 2-ig is a strengthening of primitivity.
It can also be shown that a 2-homogeneous group has the 2-ig property. So we have another condition fitting between primitive and 2-homogeneous.

The road closure property

We say that the transitive permutation group G on Ω has the road closure property if the following holds: for any orbit O of G on 2-sets, and any proper block of imprimitivity B for the action of G on O, the graph with vertex set Ω and edge set $O \setminus B$ is connected.

Clearly this implies that all the orbital graphs (the minimal non-trivial G-invariant graphs) are connected; this is equivalent to primitivity of G.

So the question: *Which groups have the road closure property?* is a question about primitive groups.

The name comes from the thought that if (Ω, O) is a connected road network, and if workers come and dig up all the roads in a block of imprimitivity, the network remains connected.

Road closure and idempotent generation

The reason why we are interested in the road closure property is given by the following theorem:

Theorem

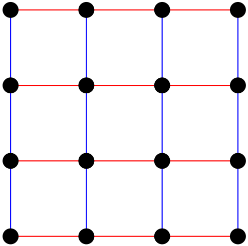
For a permutation group G on Ω, the following are equivalent:

- *for any map f of rank 2, the semigroup $\langle G, f\rangle \setminus G$ is idempotent-generated (that is, G has the 2-ig property);*
- *G has the road closure property.*

We saw already that 2-ig is stronger than 2-ut; this agrees with our observation that the road closure property is stronger than primitivity.

An example

An example of a primitive group which fails to have the road closure property is the automorphism group of the square grid graph: this is primitive for $m > 3$ (but of course not basic, since the grid is a Cartesian structure).



The automorphism group is transitive on the edges of this graph, and has two blocks of imprimitivity, the horizontal and vertical edges. If it is a road network, and if all the blue edges are closed, the network is disconnected: it is no longer possible to travel between different horizontal layers.

Non-basic groups

Theorem

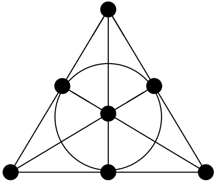
A non-basic group fails to have the road closure property.

This is similar to the example we just saw. A non-basic group preserves a Hamming graph; an orbit on edges of this graph has m blocks of imprimitivity (where m is the dimension of the Cartesian structure), and removing one of them separates the $(m-1)$-dimensional "slices".

So we need to consider only basic groups. By the O'Nan–Scott Theorem, these are affine, diagonal, or almost simple.

Another example

Not all basic groups have the road closure property.



Let G be the group of automorphisms and dualities of the Fano plane (shown), acting on the set of flags of the plane. The action is primitive. Take the orbital graph in which two flags are joined if they share a point or a line. The edges fall into two types (depending on whether the common element is a point or a line), forming blocks of imprimitivity. Removing one block, say flags sharing a point, means from a given flag we are restricted to its line.

A general result

This example extends to the following theorem.

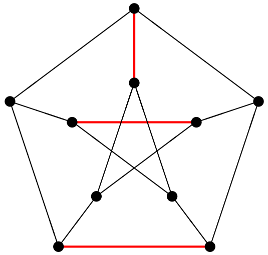
Theorem

Let G be a primitive group which has an imprimitive subgroup of index 2. Then G does not have the road closure property.

The proof is like the one just seen.

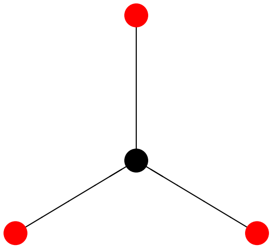
A positive example

The automorphism group of the Petersen graph (S_5 acting on 2-sets) has the road closure property. A block of imprimitivity is shown in red; its removal leaves a connected graph.



One final class

Only one further class of groups failing to have the Road Closure Property is known. These are the groups with socle $P\Omega^+(8, q)$ containing the triality automorphism, acting on the cosets of the parabolic subgroup corresponding to the three leaves of the D_4 Coxeter–Dynkin diagram.



The Road Closure Conjecture

Conjecture

Let G be a basic primitive permutation group, which has no imprimitive subgroup of index 2, and is not one of the triality examples described above. Then G has the road closure property.

This conjecture has been proved in several cases:

- 2-homogeneous groups;
- groups of prime or prime squared degree;
- symmetric or alternating groups acting on k-sets;
- groups of degree at most 130.

Partition homogeneity

I conclude with a couple of related topics.

Question

Which permutation groups G have the property that

$$\langle G, f \rangle \setminus G = \langle S_n, f \rangle \setminus S_n$$

for all maps f with $\mathrm{rank}(f) = k$?

λ-homogeneity

Let λ be a partition of the positive integer n: this means that λ is a non-increasing sequence of positive integers whose sum is equal to n.
Let $|\Omega| = n$. The shape of a partition P of Ω is the list of cardinality of the parts of P in non-increasing order. Now we say that G is λ-homogeneous if it acts transitively on the set of partitions of shape λ.
The maps of rank k in $\langle S_n, f\rangle$ are those whose kernel has the same shape as that of f and whose image has the same cardinality as that of f.
So G has the property of the preceding slide if and only if it is k-homogeneous and λ-homogeneous.

Homogeneity and transitivity

Note that a similar concept, λ-transitive, related to λ-homogeneous much as k-transitive is to k-homogeneous, was introduced by Martin and Sagan. (This requires G to be transitive on *ordered* partitions of shape λ.)

The λ-homogeneous permutation groups have been classified by André, Araújo and Cameron, and the problem posed above was solved. The λ-homogeneous groups were independently classified by Dobson and Malnič.

For $\lambda = (1, 1, \ldots, 1)$, every permutation group on Ω is λ-homogeneous, but only the symmetric group is λ-transitive. So we ignore this case.

λ-homogeneous, not λ-transitive

Theorem

Suppose that λ is a partition of n, not $(1, 1, \ldots, 1)$, and G a permutation group which is λ-homogeneous but not λ-transitive. Then one of the following happens:

- *G is intransitive; then G fixes one point and acts as S_{n-1} or A_{n-1} or one of finitely many other groups on the remainder, and λ has all parts equal; or*
- *G is transitive, and either $\lambda = (n - t, 1, \ldots, 1)$ and G is t-homogeneous but not t-transitive, or G is one of finitely many exceptions.*

The largest exception is M_{24} ($n = 24$), with $\lambda = (3, 2, 1, \ldots, 1)$.

λ-transitive

In this result we exclude $G = S_n$ and $G = A_n$. We say that (G, λ) is standard if, for some $t \leq n/2$, we have G is t-homogeneous, the largest part of λ is $n - t$, and the stabiliser of a t-set in G is μ-transitive on it, where μ consists of the remaining parts of λ. (It is easy to reduce the classification of such groups to that of t-transitive groups.)

Theorem

Let $G \neq S_n$ or A_n, $\lambda \neq (n)$. Then G is λ-transitive if and only if (G, λ) is standard.

The last two theorems give the classification of groups G satisfying

$$\langle G, f \rangle \setminus G = \langle S_n, f \rangle \setminus S_n$$

for all maps f with $\mathrm{rank}(f) = k$.

Normalizing groups

Question

Which groups G satisfy

$$\langle G, f \rangle \setminus G = \langle g^{-1} f g : g \in G \rangle$$

for all non-permutations f?

Such groups are called normalizing groups. They have been determined: we have the symmetric and alternating groups, the trivial group, and finitely many others.
The next step in this direction would be to classify the groups for which the above semigroups are equal for all maps f of given rank k (the k-normalizing groups).

Automorphisms

Perhaps the single most surprising fact about finite groups is the following.

Theorem

The only symmetric group (finite or infinite) which admits an outer automorphism is S_6.

An outer automorphism of a group is an automorphism not induced by conjugation by a group element. In the case of symmetric groups, the group elements are all the permutations, and so an outer automorphism is one which is not induced by a permutation.

The outer automorphism of S_6 was known in essence to Sylvester; it arguably lies at the root of constructions taking us to the Mathieu groups M_{12} and M_{24}, the Conway group Co_1, the Fischer–Griess Monster, and the infinite-dimensional Monster Lie algebra.

Sylvester's construction

Begin with $A = \{1, \ldots, 6\}$, so $|A| = 6$. A duad is a 2-element subset of A; so there are 15 duads. A syntheme is a set of three duads covering all the elements of A; there are also 15 synthemes. Finally, a total (or synthematic total) is a set of five synthemes covering all 15 duads. It can be shown that there are 6 totals. Let B be the set of totals.

Then any permutation on A induces permutations on the duads and on the synthemes, and hence on B; this gives a map from the symmetric group on A to the symmetric group on B which is an outer automorphism of S_6.

This outer automorphism has order 2 modulo inner automorphisms. For any syntheme lies in two totals, so we can identify synthemes with duads of totals; any duad lies in three synthemes covering all the totals, so we can identify duads with synthemes of totals; and finally, each element of A lies in 5 duads whose corresponding synthemes of totals form a total of totals!

Automorphisms of transformation semigroups

Sullivan proved the following theorem 40 years ago:

Theorem

A finite transformation semigroup S containing all the rank 1 maps has the property that all its automorphisms are induced by permutations.

Corollary

Let S be a transformation semigroup which is not a permutation group, whose group of units is a synchronizing permutation group. Then $\mathrm{Aut}(S)$ is contained in the symmetric group; that is, all automorphisms of S are induced by conjugation in its normaliser in the symmetric group.

Can we weaken "synchronizing" to "primitive" here?

A small step

If G is not synchronizing, the smallest possible rank of an element in a non-synchronizing monoid with G as its group of units is 3.

Theorem

Let S be a transformation semigroup containing an element of rank at most 3, and whose group of units is a primitive permutation group. Then the above conclusion holds: all automorphisms of S are induced by conjugation in its normaliser in the symmetric group.

We must reconstruct the points of Ω from the images and kernels of maps of rank 3 in a way which is invariant under automorphisms of S. For example, consider the images, which are maximal cliques in a graph Γ with $S \leq \mathrm{End}(\Gamma)$. No two such cliques can intersect in two points; we distinguish pairs of cliques intersecting in a point from disjoint pairs of cliques by properties of the idempotents.

… for your attention.