# Permutation Groups and Transformation Semigroups
# Lecture 4: Regularity

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Our aim in this section is to consider classes of transformation semigroups $S \leq T_n$, and investigate properties such as regularity. There is a permutation group associated with $S$ in one of two possible ways: either $S$ contains permutations, in which case $S \cap S_n$ is a permutation group; or the *normaliser* of $S$ in the symmetric group, the set

$$\{g \in S_n : (\forall s \in S) g^{-1} s g \in S\}$$

is a permutation group. In either case, as we will see, the group $G$ influences the structure of $S$.

Much as in the last chapter, we look first at semigroups of the form $\langle G, f \rangle$, where $G \leq S_n$ and $f \in T_n \setminus S_n$. If we could understand these semigroups, then we could proceed to add two or more non-permutations.

## 1 The problem

The material here grew from a theorem of Araújo, Mitchell and Schneider.

**Theorem 1.1** *Let $G$ be a permutation group on $\Omega$, with $|\Omega| = n$. Suppose that, for any map $f$ on $\Omega$ which is not a permutation, the semigroup $\langle G, f \rangle$ is regular. Then either $G$ is the symmetric or alternating group on $\Omega$, or one of the following occurs:*

*(a)* $n = 5$, $G = C_5$, $C_5 \rtimes C_2$, *or* $C_5 \rtimes C_4$;

*(b)* $n = 6$, $G = \mathrm{PSL}(2,5)$ *or* $\mathrm{PGL}(2,5)$;

*(c)* $n = 7$, $G = \mathrm{AGL}(1,7)$;

*(d)* $n = 8$, $G = \mathrm{PGL}(2,7)$;

*(e)* $n = 9$, $G = \mathrm{PGL}(2,8)$ *or* $\mathrm{P\Gamma L}(2,8)$.

(Don't worry too much about the groups in the list: the point is that, apart from finitely many exceptions, $G$ must be a symmetric or alternating group.)

In this section we explore some extensions of this result. Suppose that, instead of asking that $\langle G, f\rangle$ is regular for all choices of $f$, we require it only in some special cases, for example, all maps $f$ of some given rank $k$, or all maps $f$ whose image is a given $k$-element subset. The ultimate result would be a characterisation of all pairs $(G, f)$ for which $\langle G, f\rangle$ is regular (but we are some way from such a result now).

## 2 Multiple transitivity and homogeneity

As we saw in the first lecture, the Classification of Finite Simple Groups (CFSG) has the consequence that, for $t \geq 2$, the finite $t$-transitive groups are all known explicitly. The lists (other than symmetric and alternating groups) are finite for $t = 4,5$ and infinite for $t = 2,3$.

Much earlier, Livingstone and Wagner had investigated the relationship between $t$-homogeneity and $t$-transitivity. (We remarked earlier that $t$-transitivity implies $t$-homogeneity.) Note that a group of degree $n$ is $t$-homogeneous if and only if it is $(n-t)$-homogeneous; so, in considering these groups, we may assume that $t \leq n/2$. Now Livingstone and Wagner proved the following theorem by elementary methods:

**Theorem 2.1** *Suppose that $t \leq n/2$, and let $G$ be $t$-homogeneous of degree $n$. Then*

*(a) $G$ is $(t-1)$-homogeneous;*

*(b) $G$ is $(t-1)$-transitive;*

*(c) if $t \geq 5$, then $G$ is $t$-transitive.*

Part (a), in particular, is short and elegant, using simple facts about the character theory of the symmetric group. The permutation character on $(t-1)$-sets is contained in the character on $t$-sets, from which it follows easily that a group transitive on $t$-sets must be transitive on $(t-1)$-sets. This character-theoretic argument can be translated into elementary combinatorics.

Subsequently, Kantor determined all the $t$-homogeneous but not $t$-transitive groups for $t = 2,3,4$.

## 3 The universal transversal property

Suppose that $G$ is a permutation group, $f$ a transformation, and $fg_1fg_2f\cdots g_rf = f$. Then, for each $i$, $fg_if$ has the same rank as $f$. This implies that $g_i$ must map the image of $f$ to a section (transversal) for the kernel of $f$.

Which permutation groups $G$ have the property that, for any map $f$ of rank $k$, the element $f$ is regular in $\langle G, f\rangle$? Since the kernel and image of such a map $f$ are an arbitrary $k$-partition and an arbitrary $k$-subset, we see that a necessary condition is that $G$ has the following *k-universal transversal property*:

> For any $k$-set $A$ and any $k$-partition $P$, there is an element $g \in G$ such that $Ag$ is a transversal for $P$.

So the first question we have to consider is the classification of groups with the $k$-universal transversal property (or $k$-ut property, for short).

It turns out that this property has much stronger consequences. The implication we saw above reverses, and more besides:

**Theorem 3.1** *Given $k$ with $1 \leq k \leq n/2$, the following conditions are equivalent for a subgroup $G$ of $S_n$:*

*(a) For any rank $k$ map $f$, $f$ is regular in $\langle G, f\rangle$.*

*(b) For any rank $k$ map $f$, $\langle G, f\rangle$ is regular (this means that all its elements are regular).*

*(c) For any rank $k$ map $f$, $f$ is regular in $\langle g^{-1}fg : g \in G\rangle$.*

*(d) For any rank $k$ map $f$, $\langle g^{-1}fg : g \in G\rangle$ is regular.*

*(e) $G$ has the $k$-universal transversal property.*

The equivalence of (a) and (c) has been known for some time; but the equivalence of these two conditions with (b) and (d) is a bit of a surprise. The semigroup $\langle G, f\rangle$ usually contains elements with rank smaller than $k$; in order to show that these are regular, we need to know that $G$ has the $l$-universal transversal property, for all $l < k$:

**Theorem 3.2** *For $2 \leq k \leq n/2$, the $k$-ut property implies the $(k-1)$-ut property.*

This is reminiscent of the first part of the Livingstone–Wagner theorem. However, there seems to be no purely combinatorial proof that the $k$-ut property implies the $(k-1)$-ut property for $k \leq n/2$. So we have to make a long detour, which comes close to giving a complete classification of these groups for $k > 2$.

Which permutation groups have the $k$-ut property? In one case, the answer is simple (but shows that there is no hope of a classification):

**Proposition 3.3** *A permutation group has the* 2*-ut property if and only if it is primitive.*

This is a nice characterisation of primitivity; it depends on facts about the orbital graph which I will prove in the final lecture.

For larger values of $k$, we begin to get some hold on the group. Let us say that, if $l \leq k$, a permutation group $G$ is $(l,k)$-*homogeneous* if, for any $l$-set $A$ and $k$-set $B$, there exists $g \in G$ with $Ag \subseteq B$. If $l = k$, this is just $k$-homogeneity as previously defined.

Now observe that

> If $G$ has the $k$-ut, then $G$ is $(k-1,k)$-homogeneous.

For, given $A$ and $B$ as in the definition, take the $k$-partition $P$ which has the elements of $A$ as singleton parts and one part including everything else; then a $k$-set is a transversal for $P$ if and only it contains $A$. So, if $G$ has $k$-ut, there exists $B$ such that $Bg \supseteq A$; now the inverse of $g$ satisfies $Ag^{-1} \subseteq B$.

Also, there is a close connection between $(k-1,k)$-homogeneity and $(k-1)$-homogeneity. Certainly the second of these properties implies the first. In addition, we have

> There is a function $f$ such that, if $G$ is $(k-1,k)$-homogeneous of degree $n \geq f(k)$, then $G$ is $(k-1)$-homogeneous.

For this, we take $f(k)$ to be the Ramsey number $R_{k-1}(k,k)$. Suppose that $n \geq R_{k-1}(k,k)$ and $G$ is not $(k-1)$-homogeneous; colour the $(k-1)$-sets in one orbit red, and the remaining ones blue. The inequality on $n$ implies that there is a monochromatic $k$-set; if it is red, then no blue $(k-1)$-set can be mapped inside it by $G$, and *vice versa*.

Now the analysis involves showing that, with just five exceptions (with degrees 5, 7 and 9), a $(k-1,k)$ homogeneous group of degree $n$, with $k \leq n/2$, is $(k-1)$-homogeneous. The proof involves showing, by mostly combinatorial arguments, that such a group must be 2-transitive, and then invoking the classification of the 2-transitive groups (a consequence of CFSG). Now Theorem 3.2 follows from this, since a $(k-1)$-homogeneous group obviously has the $(k-1)$-ut property.

The permutation groups which are $(k-1,k)$-homogeneous, and those with the $k$-universal transversal property, have been almost completely classified; a few stubborn families of groups (including the Suzuki groups for $k=3$) are still holding out.

# 4 The existential transversal property

In this section we look at a more general problem. Instead of requiring regularity of $\langle G, f\rangle$ for all $f$ of rank $k$, we only ask it for those maps with a given image.

> Given $k$ with $1 \le k \le n/2$, for which permutation groups $G$ of degree $n$ and $k$-subsets $A$ of $\Omega$ is it the case that, for all maps $f$ with $\mathrm{Im}(f) = A$, the element $f$ is regular in $\langle G, f\rangle$?

For this problem, where we fix the image rather than asking about all maps with image of size $k$, a weaker property than the $k$-ut is required. We say that $G$ has the *$k$-existential transversal property*, or $k$-et property for short, if there exists a $k$-subset $A$ such that, for any $k$-partition $P$, there is an element $g \in G$ such that $Ag$ is a transversal for $P$. We call $A$ a *witnessing st* for the $k$-et property.

Work has begun on groups with the $k$-et property. It is hampered by the fact that $k$-et does not imply $(k-1)$-et for $1 < k \le n/2$: there are two 3-transitive groups of degree 16 which satisfy 4-et and 6-et but not 5-et. Also, the connection with homogeneity is not so straightforward. For example, the Mathieu group $M_{24}$ (which is 5-transitive but not more) has the 7-et property.

Another issue is that the $k$-et property does not imply transitivity. Fortunately, however, it is possible to determine completely the intransitive groups with the property for $2 < k < n$: such a group must fix a point and act $(k-1)$-homogeneously on the remaining points. So we may assume that our group $G$ is transitive.

However, using CFSG and quite a bit of effort, it has been possible to show:

**Theorem 4.1** *Suppose that $8 \le k \le n/2$. Then a transitive permutation group of degree $n$ with the $k$-et property is the symmetric or alternating group.*

The example $M_{24}$ shows that 8 is best possible in this theorem; but probably $M_{24}$ is the only further example for the 7-et property.

Here is a short account of the proof, giving the main techniques used.

Suppose that $G$ has the $k$-et property and let $A$ be a witnessing set. First note that $A$ contains a representative of every $G$-orbit on $(k-1)$-sets. For, if $B$ is a $(k-1)$-set, let $P$ be the partition which has the singletons of $B$ as parts and a single part containing everything else. Then $A$ can be mapped to a transversal for $P$, that is, a $k$-set containing $B$.

In particular, this means that $G$ has at most $k$-orbits on $(k-1)$-sets, and so

$$|G| \geq \binom{n}{k-1} \Big/ k.$$

We call this the *order bound*, and return to it shortly. We note that the right-hand side of the order bound gets stronger as $k$ increases (for $k \leq n/2$); so, if $G$ fails the bound for some $k$, then it fails for all larger $k$. Indeed, this bound has been improved by essentially a factor of 2 by Wolfram Bentz very recently.

The other main technique is that, if $G$ is a group of automorphisms of some combinatorial structure, we can often find two $(k-1)$-subsets of that structure which cannot "coexist" inside a $k$-set, which would contradict the above property of the witnessing $k$-set. For example, suppose that $k=4$, and that $G$ is imprimitive, with at least three blocks each of size at least 3. Then a 3-subset of a block and a 3-set containing a point from each of three distinct blocks cannot coexist. Pursuing this argument a little further, we conclude that, for $k \geq 4$, $G$ must be primitive.

The argument above gives us a lower bound for the order of $G$. There are also upper bounds for orders of primitive groups – the best of these is derived from CFSG. Combinatorial arguments also have their place here.

For example, suppose that $G$ is one of the "large" primitive groups, $S_m$ or $A_m$, in its action on 2-sets. As we saw in the preceding chapter, $G$ preserves a graph (the line graph of $K_m$) which contains a clique of size $m-1$ and an independent set of size $\lfloor m/2 \rfloor$, which cannot coexist inside a set of size less than $m-2+\lfloor m/2 \rfloor$; so $k$ must be at least this value. Now it is easy to see that there are more than $k$ orbits on $(k-1)$-sets (these orbits correspond to graphs with $k-1$ edges).