The ADE affair

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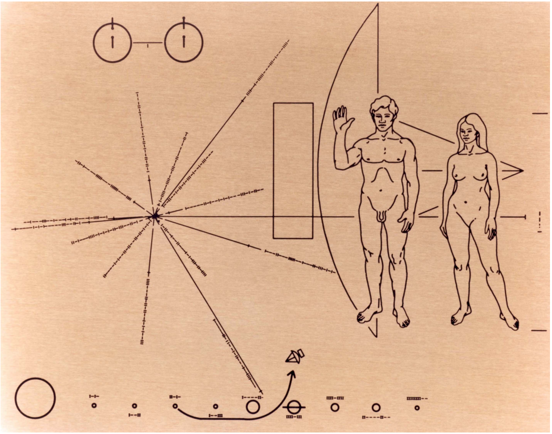
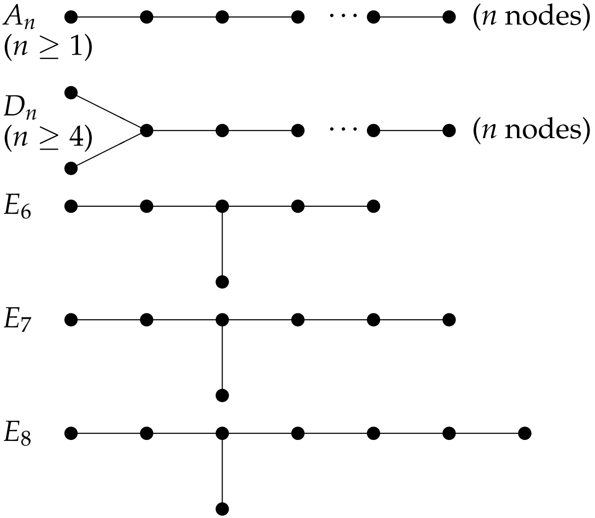
Calling card

What should humanity send as a calling card for extraterrestrial civilisations?
The Pioneer engineers decided that this would do:



Francis Buekenhout had another idea:



A modern Hilbert problem?

In an article on "Problems of present day mathematics" in an AMS Symposium on *Mathematical Developments arising from Hilbert Problems*, V. Arnold wrote,
The Coxeter–Dynkin graphs A_k, D_k, E_k appear in many independent classification theorems. For instance:

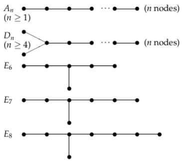
- … the platonic solids (or finite orthogonal groups in Euclidean 3-space).
- … the categories of linear spaces and maps.
- … the singularities of algebraic hypersurfaces with a definite intersection form of the neighboring smooth fiber.
- … critical points of functions having no modules.
- … the Coxeter groups generated by reflections, or of Weyl groups with roots of equal length.

Arnold forgot to mention one of the best-known occurrences, in the classification of simple Lie algebras and Lie groups.
Since he wrote, several more such classifications have arisen, including

- graphs with least eigenvalue -2;
- instantons and asymptotically locally Euclidean spaces;
- cluster algebras.

Buekenhout's view was that, even if an alien civilisation had very different mathematics to ours, chances are that they would have come up with at least some of the areas in which the ADE diagrams occur.
I will say a bit about all this.

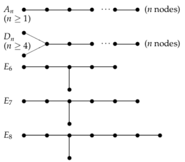
The extended diagrams



Closely related to the ADE diagrams are the so-called extended diagrams. In each case the extension adds one vertex, as follows:

- for A_n, the new vertex is joined to both ends of the path, forming an $(n+1)$-cycle;
- for D_n, it creates a fork at the other end of the diagram, or a $K_{1,4}$ if $n = 4$;
- for E_n, it extends one of the arms, so that the numbers of vertices on the arms are $(3, 3, 3)$ $(n = 6)$, $(2, 4, 4)$ $(n = 7)$, or $(2, 3, 6)$ $(n = 8)$.

What are these diagrams?



The spectrum of a graph is the spectrum of its adjacency matrix. J. H. Smith proved the following theorem.

Theorem

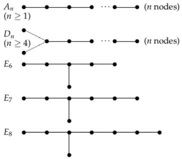
- *The connected graphs with greatest eigenvalue strictly less than* 2 *are precisely the ADE diagrams.*
- *The connected graphs with greatest eigenvalue equal to* 2 *are precisely the extended ADE diagrams.*

Although this theorem was proved in 1969, it is in some sense implicit in the classification of simple Lie algebras over $\mathbb{C}$.

Sketch proof

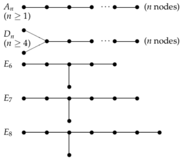
Here is an outline proof of the first part.
It is easy to see that the extended ADE diagrams have greatest eigenvalue 2, as we will see shortly. So a connected graph whose greatest eigenvalue is less than 2 cannot contain any of these.
In particular, it does not contain $\tilde{A}_n$ (a cycle), and so is a tree; it does not contain $\tilde{D}_n$, and so has at most one branchpoint, such a point having valency at most 3; and it does not contain $\tilde{E}_n$, and so the lengths of the three arms are restricted to the appropriate values.

Polyhedra and tessellations



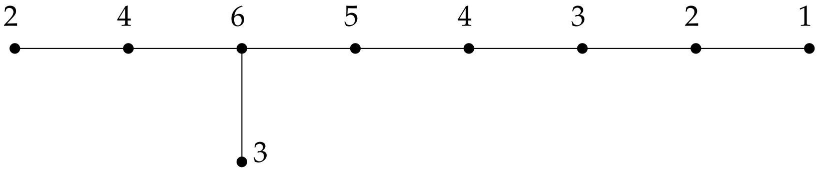
The numbers of vertices on the arms of the E_n diagrams are $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ for $n = 6, 7, 8$ respectively. This should remind you of the regular polyhedra in 3-space.
The corresponding numbers for the extended diagrams are, as we saw, $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ respectively, corresponding to the regular tessellations of the Euclidean plane by triangles, squares and hexagons.
These are not accidental; we will return to them later.

A detour

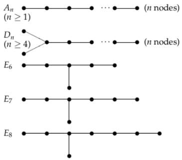


By the Perron–Frobenius Theorem, a connected graph has a unique eigenvector (up to scalar multiple) corresponding to the greatest eigenvalue; this eigenvector has all its entries positive.
This means that we can label the vertices of the extended ADE diagrams with positive numbers so that the sum of labels on the neighbours of a vertex v is twice the label on v.
It is a simple exercise to write down such a labelling. See the next slide for $\tilde{E}_8$.

An eigenvector



Finite rotation groups



As is well known, the list of finite groups of rotations in 3 dimensions (or finite subgroups of $SO_3(\mathbb{R})$ is as follows:

- cyclic groups $C_n = \langle x | x^n = 1 \rangle$;
- dihedral groups $D_n = \langle x, y | x^n = y^2 = (xy)^2 = 1 \rangle$;
- the tetrahedral group $\langle x, y : x^2 = y^3 = (xy)^3 = 1 \rangle$;
- the octahedral group $\langle x, y : x^2 = y^3 = (xy)^4 = 1 \rangle$;
- the icosahedral group $\langle x, y : x^2 = y^3 = (xy)^5 = 1 \rangle$.

The correspondence with the ADE diagrams is clear.

Binary rotation groups

There is a two-to-one homomorphism from the special unitary group $SU_2(\mathbb{C})$ to the rotation group $SO_3(\mathbb{R})$.
The inverse image of each finite rotation group $G \leq SO_2(\mathbb{R})$ is a double cover $\tilde{G}$ of G in $SU_2(\mathbb{C})$, a group with a centre Z of order 2 such that $\tilde{G}/Z = G$. (Note, incidentally, that Z contains the unique involution in $\tilde{G}$.)
Each of these groups comes with a "natural" two-dimensional unitary representation ρ, which is in fact self-dual (implying that $\rho \otimes \rho$ contains the trivial representation).

McKay's observation

From the preceding, we see that there is a graph structure on the set of irreducible complex representations of $\tilde{G}$: an edge joins the representations σ_1 and σ_2 if σ_2 is a constituent of the representation $\sigma_1 \otimes \rho$.
Our earlier observation shows that the graph is undirected.
If we label each node with the degree of the corresponding representation σ, we see that the labels of the neighbours of σ add up to the degree of $\sigma \otimes \rho$, that is, twice the degree of σ.
So we have a positive eigenvector with eigenvalue 2.
We conclude that the graph is an extended ADE diagram.
Indeed, the correspondence agrees with the one we just observed for the rotation groups.

A story

When I was a fellow of Merton College, Oxford, Peter Kronheimer was an undergraduate at the college. In his final year, he came to ask me about McKay's observation.
After obtaining a very good First in his finals, he did a DPhil under the supervision of Michael Atiyah. In his thesis, he used this observation to construct a family of instantons, or asymptotically locally Euclidean (ALE) spacetimes.
I am afraid I am not competent to tell you about this, but I am very happy to have had a part in the story!

Another story

The moral of the ADE classification is that, if you solve a classification problem to which the answer turns out to be “two infinite familes and three sporadic examples”, then chances are that your problem is connected with the ADE affair.

In the 1970s, two of my collaborators, Jaap Seidel and Jean-Marie Goethals, were working on graphs with smallest eigenvalue -2.

I visited Jaap in Eindhoven; he explained to me that he had solved a technical problem that arose, and found one infinite family and three sporadic examples.

The next day, we visited Jean-Marie in Brussels, who explained that he had also succeeded, and found one infinite family and three sporadic examples.

On comparing their work, each of them had missed a different infinite family. So it was clearly an ADE problem, as I will now describe.

Root systems

A root system is a very geometric object. It is a finite set R of non-zero vectors in Euclidean space $\mathbb{R}^n$ with the properties

- If $v \in R$, then for any real number c, $cv \in R \Leftrightarrow c = \pm 1$.
- For all $v, w \in R$, $2(v \cdot w)/(v \cdot v) \in \{0, \pm 1, \pm 2, \pm 3\}$.
- For all $v \in R$, the reflection σ_v in the hyperplane perpendicular to v maps R to itself.

Note that

- $2(v \cdot w)/(v \cdot v) \times 2(v \cdot w)/(w \cdot w) \leq 4$ by Cauchy–Schwarz, so if there are roots of different lengths then the ratio of their squared lengths is 2 or 3.
- $\sigma_v(w) = w - 2(v \cdot w)/(v \cdot v)v$, an integer linear combination of v and w. From this, it can be shown that R spans a lattice (a root lattice).

Roots of constant length

It is clear that the vectors of fixed length in a root system form themselves a root system; so it is enough to classify these, and then piece together at most two such for the general classification.

If all the roots have the same length, then the angles between two roots are 60°, 90°, 120° or 180°.

It can be shown that there is a basis consisting of roots with non-positive inner products. If G is the Gram matrix, then (after normalisation) we have $G = 2I - A$, where A is the adjacency matrix of a graph. Since G is positive definite, A has greatest eigenvalue less than 2.

Thus, if the root system is indecomposable (that is, the graph is connected), it is an ADE diagram. Moreover, the graph determines the root system.

From the icosahedron to E_8

Is there a direct connection between the 3-dimensional rotation groups and the root systems, underlying the McKay correspondence? A beautiful construction was found recently by Pierre Dechant.
This uses the *Clifford algebra*, a deformation of the exterior algebra: we consider only the case of real vector spaces V with a positive definite inner product. The multiplication in the Clifford algebra is given by

$$xy = x \cdot y + x \wedge y,$$

where the inner and outer (exterior) product are the symmetric and antisymmetric parts of the Clifford algebra product. The dimension of the Clifford algebra is the same as that of the exterior algebra on V, namely 2^n, where $n = \dim(V)$.

We take $n = 3$, so that the dimension of the Clifford algebra is 8. A short calculation shows that the reflection in the hyperplane perpendicular to the unit vector a is given by

$$\sigma_a(v) = -ava.$$

It is the same for a and $-a$, so each 3-dimensional reflection corresponds to two vectors in the Clifford algebra.

The group of reflections and rotations of the icosahedron has order 120, and so it lifts to a set of 240 vectors in $\mathbb{R}^8$. These form the E_8 root system.

There is much more to the story, but no time to tell it here …

The remaining root systems

For roots of different lengths, it is not hard to show:

- If the ratio of squared lengths is 3, then the dimension is 2, and roots of each length have type A_2 (vertices of a hexagon): this gives the root system G_2.
- If the ratio of squared lengths is 2, then either
  - one root system has type D_n and the other is an orthonormal basis, giving types B_n or C_n (depending on which roots are longer); or
  - the two root systems have type D_4, and together give type F_4.

Lie algebras

To classify simple Lie algebras over $\mathbb{C}$, one develops their structure theory to the point where one finds an indecomposable root system in the dual space of the Cartan subalgebra.
Then appeal to the classification of root systems, and show that the root system uniquely determines the algebra.
The dimension of the Lie algebra arising from the root system R is $\dim(R) + |R|$ (the first term for the Cartan algebra, and the second for the root subspaces). For E_8 we obtain $8 + 240 = 248$.

Star-closed sets

To classify the graphs with least eigenvalue -2, we reverse the procedure.

Let A be the adjacency matrix of such a graph. Then $A + 2I$ is positive semidefinite, and so is the matrix of inner products of a set of vectors in $\mathbb{R}^n$, where n is the comultiplicity of -2 as an eigenvalue. Clearly any two of these vectors make an angle 90° or 60°.

Now a geometric argument shows that a set of lines through the origin in $\mathbb{R}^n$, in which any two lines make angle 90° or 60°, and is maximal with respect to this property, is star-closed: that is, if two lines in the set make angle 60°, then the third line in their plane at 60° to both is also in the set.

Taking vectors of fixed length on the lines in such a star-closed set, we obtain a root system!

The classification

Thus, graphs with least eigenvalue -2 are "contained" in a root system of type ADE.
It can be shown that the A_n root system is contained in D_{n+1}.
Moreover, graphs represented in the root system D_{n+1} are what Alan Hoffman defined as generalised line graphs.
So we have

Theorem

A connected graph with smallest eigenvalue -2 (or greater) is either a generalized line graph, or one of finitely many exceptions (all these exceptions being represented in the root system E_8).

The paper containing this theorem (with Goethals, Seidel, and Ernest Shult) is one of my most cited.

Abelian unipotent groups

This theorem was used to classify the abelian subgroups generated by root subgroups in groups of Lie type.
It follows from the commutation relations for such subgroups that, if two of them commute, then the corresponding roots make an angle at most 90°.
So subgroups with the required property are classified by the graphs with least eigenvalue -2, and the preceding theorem applies.

And more . . .

I have not told you the whole story. There are connections with algebras of finite representation type, with critical points of smooth functions, and with cluster algebras (which come up in the theories of Poisson algebras and totally positive matrices). Indeed, when Fomin and Zelevinsky invented cluster algebras, they didn't at first know that the finite dimensional ones fitted the ADE classification, but discovered this later.

Cluster algebras: a taster

Consider the sequence given by the recurrence

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}.$$

It is easy to see that it returns to its initial value after five steps.
If the first two terms are $1, 1$, then the sequence runs
$1, 1, 2, 3, 2, 1, 1, \ldots$.
However the related recurrence

$$x_{n+3} = \frac{1 + x_{n+1}x_{n+2}}{x_n}$$

starting $1, 1, 1$, runs $1, 1, 1, 2, 3, 7, 11, 26, 41, 97, 153, 362, 571, \ldots$, and grows forever. This sequence has many interesting interpretations, including the denominators of the continued fraction convergents to $\sqrt{3}$.

Each of these examples is associated with a cluster algebra associated with a quiver (oriented graph). In the first case it is an orientation of A_2 (a single edge); the second case is an oriented triangle, so not an ADE diagram. This agrees with the finiteness theorem of Fomin and Zelevinsky.
Note too that, despite the denominators in the recurrence, the sequences generated consist of integers. This is explained by the theory.
But note that the period 5 in the first case seems to have nothing to do with the root system, or reflection group, or anything else, traditionally associated with A_2. There are groups here too, which are not the Coxeter groups of the appropriate types. For the second example, we get the group of rotations of the icosahedron . . .

Conclusion

So, have mathematicians succeeded in answering Arnold’s question, or have we simply added further layers of mystery?

You decide . . .