Combining the cycle index and the Tutte polynomial?

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Selections

Students often meet the following table in elementary probability or combinatorics courses. In how many ways can I choose n objects from a set of k objects?

	Order significant	Order not significant
With replacement	k^n	$\binom{k+n-1}{n}$
Without replacement	$k(k-1)\cdots(k-n+1)$	$\binom{k}{n}$

Symmetry and structure

The first entry k^n is in a sense the most basic.

The rows and the columns reflect two important themes of combinatorial enumeration, namely symmetry and structure.

In the first column, we count selections as they are drawn; in the second, we count up to symmetry (where 'symmetry' means arbitrary rearrangement of the order of the draws).

In the first row, we count all selections; in the second, we place a structural restriction on the allowed selections, namely, the elements drawn must all be distinct.

In order to generalise, we note that we are counting functions from an n-set to a k-set. In the second row we count injective functions, while in the second column we count up to the action of the symmetric group S_n.

Chromatic polynomial

We begin by generalising the structural restriction.

Let Γ be a graph on n vertices. A proper k-colouring of Γ with a set C of k colours is a function from the vertex set $V\Gamma$ to the set of colours, with the property that adjacent vertices receive different colours.

The chromatic polynomial $P_\Gamma(x)$ of the graph Γ has the property that, for positive integers k, $P_\Gamma(k)$ is the number of proper k-colourings of Γ.

Thus, if Γ is the null graph (with no edges), any function is a colouring, and $P_\Gamma(x) = x^n$; if Γ is the complete graph (with an edge between each pair of vertices), then the proper colourings are just the injective functions, and

$$P_\Gamma(x) = x(x-1)\cdots(x-n+1).$$

Chromatic number, chromatic roots

The chromatic number of a graph Γ is the smallest positive integer k for which Γ has a proper colouring with k colours.

A positive integer k is a root of $P_\Gamma(x)$ if and only if k is smaller than the chromatic number of Γ.

A chromatic root is a root of a chromatic polynomial.

- There are no chromatic roots in the intervals $(-\infty, 0)$, $(0, 1)$, or $(1, \frac{32}{27}]$ (Jackson)
- Real chromatic roots are dense in $[\frac{32}{27}, \infty)$ (Thomassen)
- Complex chromatic roots are dense in $\mathbb{C}$ (Sokal)

Orbit-counting Lemma

Next we turn to symmetry.

Let G be a group of automorphisms of a graph Γ. We want to count G-orbits on the set of proper colourings of Γ. The key tool is the Orbit-counting Lemma:

Theorem
Let G act on a set X. Then the number of orbits of G on X is equal to the average number of fixed points on X of the elements of G:

$$\# \operatorname{Orbits}(G, X) = \frac{1}{|G|} \sum_{g \in G} \operatorname{fix}_X(g).$$

Said otherwise, the number of orbits is the expected number of fixed points of a random element of G.

Orbital chromatic polynomial

Let g be an automorphism of a graph Γ. Denote by Γ/g the graph obtained by shrinking every cycle of g to a single vertex. The number of k-colourings of Γ fixed by g is equal to the number of colourings of Γ/g. For a colouring is fixed by g if and only if it is constant on the cycles of g (and so induces a proper colouring of Γ/g).
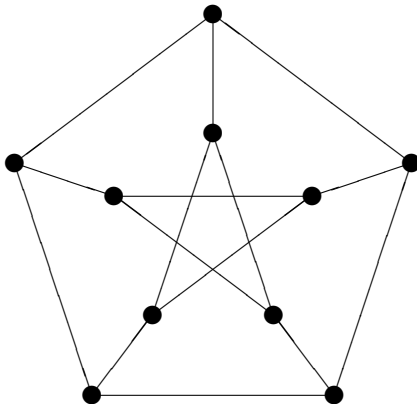
So, if G is a group of automorphisms of Γ, define the orbital chromatic polynomial of Γ and G to be

$$OP_{\Gamma,G}(x) = \frac{1}{|G|} \sum_{g \in G} P_{\Gamma/g}(x).$$

Then for positive integers k, the number of orbits of G on the k-colourings of Γ is $OP_{\Gamma,G}(k)$.

An example

I will illustrate with the Petersen graph:

For the Petersen graph, the chromatic polynomial is

$$P_\Gamma(q) = q(q-1)(q-2) \times$$
$$(q^7 - 12q^6 + 67q^5 - 230q^4 + 529q^3 - 814q^2 + 775q - 352).$$

We see that the least number of colors required for a proper coloring (the smallest q for which this is non-zero) is 3, and the number of proper colorings for q up to 10 is given in the following table:

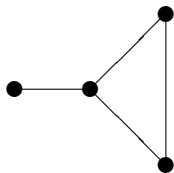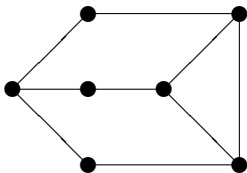| q	3	4	5	6	7	8	9	10
$P_\Gamma(q)$	120	12960	332880	3868080	27767880	144278400	594347040	2055598560

In order to calculate the orbital chromatic polynomial of the Petersen graph, we need to understand its automorphisms. (We take G to be the full automorphism group.)

A convenient representation of the Petersen graph is as follows. The vertices can be labelled with the 2-element subsets of $\{1, 2, 3, 4, 5\}$; two vertices are joined if and only if their labels are disjoint. It is clear from this description that the symmetric group S_5 acts on the graph, and in fact this is the full automorphism group.

Now the only automorphisms whose cycles contain no edges are the identity, 2-cycles, and 3-cycles on $\{1, 2, 3, 4, 5\}$. (For example, the cycle $(12, 23, 34, 45, 15)$ of the permutation $(1, 2, 3, 4, 5)$ contains an edge from 12 to 34.) There are 10 2-cycles and 20 3-cycles, and the corresponding graphs Γ/g are shown:

Now doing the calculation, we find that the orbital chromatic polynomial is

$$OP_{\Gamma,G}(q) = q(q-1)(q-2) \times$$
$$(q^7 - 12q^6 + 67q^5 - 220q^4 + 469q^3 - 664q^2 + 595q - 252)/120.$$

The values for 3 to 10 colors are

q	3	4	5	6	7	8	9	10
$OP_{\Gamma,G}(q)$	6	208	3624	36654	248234	1254120	5089392	17449788

A variation

In how many ways can we colour a graph if the order of the colours is not significant? This asks us to count partitions into independent sets.

We can't just divide the chromatic polynomial by $q!$. [WHY??]

First, we count colourings in which every colour actually appears. This is straightforward using inclusion-exclusion: we obtain

$$P_\Gamma^*(q) = \sum_{r \leq q} (-1)^{q-r} \binom{q}{r} P_\Gamma(r).$$

(Note that this is *not* a polynomial, since it is zero if $q > n$.)

Dividing this number by $q!$, we obtain the number of partitions into q independent sets.

For the Petersen graph, the numbers are

q	3	4	5	6	7	8	9	10
$P_\Gamma^*(q)/q!$	20	520	2244	2865	1435	315	30	1

Combining?

What if we don't care about the names of the colors, and also want to count up to symmetry?

The first step of what we did works fine: PIE gives us a formula for the number of orbits on q-colorings in which all the colors are used. The numbers for the Petersen graph are

q	3	4	5	6	7	8	9	10
$OP^*_{\Gamma,G}(q)$	6	184	2644	17910	60690	105840	90720	30240

But we cannot simply divide these numbers by $q!$ to get the number of orbits on partitions. This is because it is possible that a permutation of the parts of a partition can be realised by applying a symmetry, so we would be undercounting. Indeed, the numbers in the table are not all divisible by $q!$.

I don't know a mechanical method of finding the number of orbits of G on partitions of Γ into q independent sets. This would be an interesting research problem. In the case of the Petersen graph, the six orbits with $q = 3$ are indeed all the same if permutations of the colors are allowed, so the first entry in the corresponding table is 1.

Computationally, the result is:

q	3	4	5	6	7	8	9	10
$P^*_{\Gamma,G}(q)$	1	10	30	36	20	7	1	1

Notice how much smaller the numbers are!

Reciprocity theorems

I'd like to take a small detour to discuss some recent work with Jason Semeraro.

Think back to the formulae for the numbers of unordered selections of k things from n:

Without repetition: $\binom{n}{k}$; With repetition: $\binom{n+k-1}{k}$.

Both formulae are polynomials of degree k in n, and we have

$$\binom{n+k-1}{k} = (-1)^k \binom{-n}{k}.$$

This is a simple example of what Richard Stanley called a reciprocal pair of polynomials, two combinatorially-defined polynomials f and g of the same degree k, which (slightly unexpectedly) satisfy the relation

$$g(q) = (-1)^k f(-q).$$

The cycle polynomial of a permutation group

Let G be a permutation group of degree n. The cycle polynomial of G is the polynomial

$$F_G(x) = \frac{1}{|G|} \sum_{g \in G} x^{c(g)},$$

where $c(g)$ is the number of cycles of the permutation g (including fixed points). It is a polynomial of degree n, with leading coefficient $1/|G|$.

Proposition

For a positive integer q, $F_G(q)$ is the number of G-orbits on colourings of $\{1, \ldots, n\}$ with q colours.

This follows immediately from the Orbit-Counting Lemma, since a colouring is fixed by g if and only if it is constant on every cycle of g.

An example

It follows from the preceding Proposition that

$$F_{S_n}(x) = \binom{x+n-1}{n}.$$

We recognise the right-hand side as the reciprocal polynomial of $\binom{x}{n}$, which happens to be the orbital chromatic polynomial of S_n acting on the complete graph K_n (since in a proper colouring, all the colours are forced to be different).

Jason and I found many instances of the scenario described in the following question.

Question

For which permutation groups G does there exist a graph Γ such that $F_G(x)$ and $OP_{\Gamma,G}(x)$ are reciprocal polynomials?

The cycle index

The cycle index of a permutation group G of degree n is the polynomial in indeterminates $s_1, \ldots, s_n$ given by

$$Z(G; s_1, \ldots, s_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{n} s_i^{c_i(g)},$$

where $c_i(g)$ is the number of cycles of length i in the permutation g.

Its role in many combinatorial counting problems involving enumeration under group action is well-known.

The cycle polynomial we met earlier is a simple specialisation:

$$F_G(x) = Z(G; x, x, \ldots, x).$$

Other specialisations have been studied: among these are the fixed point polynomial, the generating function for the numbers of fixed points of group elements, is $Z(G; x, 1, \ldots, 1)$; and the Parker vector, used in computing Galois groups.

A problem

We saw that the cycle polynomial sometimes has a reciprocal polynomial, which is the orbital chromatic polynomial of a G-invariant graph.

Richard Stanley has suggested that there might be a reciprocal of the cycle index. I am not sure how this would work.

If it were so, then other specialisations such as the fixed point polynomial would presumably have reciprocals also.

Question

Investigate specialisations of the cycle index and study their reciprocal polynomials. Which ones have natural combinatorial interpretations?

The Tutte polynomial

The chromatic polynomial of a graph is generalized by the two-variable Tutte polynomial, which has specialisations counting many things other than colorings.

The Orbit-Counting Lemma can be systematized by the multivariable cycle index of a permutation group, which yields counts for orbits of the group on very general configurations.

Problem

Combine the two approaches, to count orbits of G on graph-theoretic objects counted by the Tutte polynomial.

I will briefly describe the situation for flows and tensions on a graph. This is taken from a paper with Bill Jackson and Jason Rudd.

Tensions and flows

Let A be a finite abelian group. Choose a fixed but arbitrary orientation of the edges of the graph Γ.

- A tension on Γ (over A) is a function from the set of arcs of Γ to A with the property that the signed sum of the values along any circuit is zero.

- A flow on Γ (over A) is a function from the set of arcs of Γ to A with the property that the signed sum of the values on the arcs through any vertex is zero.

We are interested in the numbers of nowhere-zero tensions and flows.

Cleary, reversing an arc and changing the sign of the function there does not affect the property of being a tension or flow; so the numbers of these are independent of the orientation.

Tensions

It is easy to see that any tension is obtained as follows. Choose a function Φ from the vertex set of Γ to A; now put on each arc the difference between its values at the head and the tail.

Now Φ is a proper coloring of Γ if and only if the derived tension is nowhere-zero.

There is one free choice for the value of Φ in each connected component. So the number of nowhere-zero tensions is $P_\Gamma(q)/q^c$, where c is the number of connected components of Γ, and $q = |A|$. Note that the number does not depend on the structure of A, only its order. We call $P_\Gamma(q)/q^c$ the tension polynomial of X.

Flows

It is often claimed that flows are "dual" to colorings. What they are really dual to is tensions, as we will see.

Tutte showed that the number of nowhere-zero flows does not depend on the structure of A, but only on its order; this number is a polynomial in $q = |A|$, called the flow polynomial of the graph Γ.

For planar graphs, the flow polynomial of a graph is the tension polynomial of its dual.

There are many interesting unsolved problems about roots of flow polynomials.

Orbital versions

There are orbital tension and flow polynomials associated with a graph Γ and group G of automorphisms of Γ.

However, unlike the usual polynomials, they are multivariate. We have variables x_i for $i = 0$ and for each positive integer i for which there is an element of order i in the group G.

To obtain a count for the number of orbits of G on tensions and/or flows over an abelian group A, we substitute α_i for x_i in the appropriate polynomial, where α_i is the number of solutions of $ia = 0$ for $a \in A$. Note that $\alpha_0 = |A|$ and $\alpha_1 = 1$; so if G is trivial, the result depends only on $|A|$, in agreement with Tutte's observation.

An example

The Petersen graph is a bit big, so I will consider the following graph. Edges are directed top-to-bottom, and the letters indicate the values (in an abelian group A) of the function.
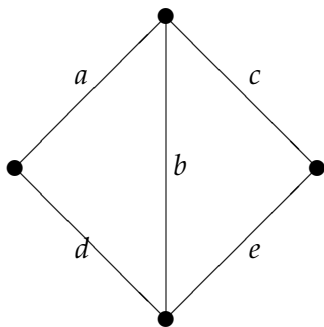
The group G consists of the identity, the left-to-right reflection r_1, the top-to-bottom reflection r_2, and their product.

Orbits on nowhere-zero tensions

Let $q = |A|$ and α_i as before.

- For a tension, we have $a + d = b = c + e$, so there are $(q-1)(q-2)^2$ non-zero tensions.
- A tension is fixed by r_1 if $a = c$. So there are $(q-1)(q-2)$ such tensions.
- A tension is fixed by r_2 if $d = -a$, $e = -c$ and hence $b = 0$. So there are no fixed tensions.
- A tension is fixed by $r_1 r_2$ if and only if $e = -a$, $d = -c$, and $b = -b$. So $2b = 0$ (α_2 choices), then there are $q-2$ choices for a, after which everything is determined. So $\alpha_2(q-1)$ fixed tensions.

So the orbital tension polynomial is

$$\frac{1}{4}(x_0 - 1)((x_0 - 1)(x_0 - 2) + x_2).$$

Orbits on nowhere-zero flows

- For a flow, we have $d = a$, $e = c$, and $a + b + c = 0$. So $(q - 1)(q - 2)$ choices.
- A flow is fixed by r_1 if $a = b$. So $c = -2a$ and we require $2a \neq 0$: $q - \alpha_2$ choices.
- A flow is fixed by r_2 if $2a = 2b = 2c = 0$. So $(\alpha_2 - 1)(\alpha_2 - 2)$ choices.
- A flow is fixed by $r_1 r_2$ if $a + e = c + d = 2b = 0$. Then $a + c = 0$, so $b = 0$, and there are no fixed flows.

So the orbital flow polynomial is

$$\frac{1}{4}((x_0 - 1)(x_0 - 2) + (x_0 - x_2) + (x_2 - 1)(x_2 - 2)).$$

Matroids

In the paper with Jackson and Rudd, we also

- combine the tension and flow polynomials into an orbital Tutte polynomial associated with a graph and a group of automorphisms, having two potentially infinite sequences of variables;
- extend the theory from graphs to the class of matroids representable over principal ideal domains (this also allows us to define an orbital weight enumerator of a linear code).

I will not attempt to describe the construction here.

Matroids and cycle index

Looking further, it is interesting to speculate that there are permutation groups which have the property that the cycle index (or some specialisation of it) has a reciprocal polynomial which is the orbital Tutte polynomial of a matroid admitting the group as automorphisms.

One class of groups for which some results have been obtained consists of those which are the hyperplane families of permutation geometries, in the sense of Michel Deza. These are the analogue, in the semilattice of partial permutations on a set, of the geometry of flats of a matroid in the Boolean lattice of subsets.

More precisely, these are the permutation groups in which the stabiliser of any finite sequence of points acts transitively on the points it moves (if any).

These groups have all been classified (by Zil'ber for rank at least 7, by complicated geometric arguments; by Maund, using the Classification of Finite Simple Groups, for all ranks greater than 1.)

Some connections between cycle index of the group and Tutte polynomial of the matroid have been established for these groups, but the situation is still not well understood.