Equitable partitions of Latin square graphs

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Equitable partitions

We have a graph Γ on the vertex set Ω; we assume that Γ is connected and is regular with valency k.

A partition $\{\Delta_1, \ldots, \Delta_r\}$ of Ω is equitable if there is a matrix $M = (m_{ij})$ such that a vertex in Δ_i has exactly m_{ij} neighbours in Δ_j.

Examples:

- The orbits of a group of automorphisms of Γ.
- The distance partition with respect to any vertex is equitable with the same matrix if and only if the graph is distance-regular.
- Many examples in finite geometry, including ovoids, spreads, and Cameron–Liebler line classes, fit into this framework.

The spectrum

Let Γ have adjacency matrix A. Let Δ be an equitable partition with matrix M. If $\mathbf{v}_i$ is the characteristic function of Δ_i, then

$$\mathbf{v}_j A = \sum \mathbf{v}_i m_{ij},$$

so the spectrum of M is contained in that of A.
M always has eigenvalue k, the principal eigenvalue, since its row sums are equal to k. We say that the partition is μ-equitable if all non-principal eigenvalues of M are equal to μ. This means that the vectors $\mathbf{v}_i$ all lie in the sum of the k- and μ-eigenspaces of A.

Perfect sets

A subset S of Ω is perfect if the partition $\{S, \Omega \setminus S\}$ is equitable; it is μ-perfect if the partition is μ-equitable.

Now easy linear algebra shows that a partition Δ is μ-equitable if and only if all but at most one part of the partition is μ-perfect.

In particular, to find all μ-equitable partitions, it suffices to find all the *minimal* μ-perfect sets.

Latin square graphs

A Latin square of order n is an $n \times n$ array with entries from an alphabet of n letters, such that each letter occurs once in each row and once in each column.

Given a Latin square L, we define the corresponding Latin square graph $\Gamma(L)$ to have as vertices the n^2 cells of the array L, two vertices joined if they lie in the same row or the same column or contain the same letter.

The eigenvalues of the adjacency matrix are $3(n-1)$ (the principal eigenvalue, with multiplicity 1); $n-3$ (with multiplicity $3(n-1)$), and -3 (with multiplicity $(n-1)(n-2)$.

First examples

Let S be the set of n cells in a row. Then $\{S, \Omega \setminus S\}$ is equitable, with matrix

$$\begin{pmatrix} n-1 & 2(n-1) \\ 2 & 3n-5 \end{pmatrix},$$

so S is $(n-3)$-perfect. Of course, the same applies to any column or letter.

What G and G did

At the International Workshop on Bannai–Ito Theory in Hangzhou, Sergey Goryainov talked about a result he had proved with his supervisor Alexander Gavrilyuk. Although phrased in terms of bilinear forms, it amounted to a complete determination of the $(n-3)$-equitable partitions (or, equivalently, the minimal $(n-3)$-perfect sets) in a particular type of Latin square graph: the Cayley table of an elementary abelian 2-group.
The result is that these are rows, columns, letters, or one more type: subsquares of order $n/2$ corresponding to subgroups of index 2 in the group.
RAB and PJC wondered whether this could be generalised …

More examples

They found two new constructions of $(n-3)$-perfect sets:
Corner sets in the Cayley tables of cyclic groups. These have shape

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Inflation Take a Latin square L_0 of order s. Replace each occurrence of letter i be a Latin square of order t in alphabet A_i, where the alphabets for different letters are pairwise disjoint; this gives a Latin square L of order $n = st$. Moreover, given an $(s-3)$-perfect set S_0 in L_0, the corresponding cells in L form an $(n-3)$-perfect set.
For example, inflating a single entry in the 2×2 Latin square gives the G–G example.

The theorem

Theorem

Let S be a minimal $(n-3)$-perfect set in the graph of a Latin square of order n. Then S is a row, a column, a letter, or an inflation of a corner set.

So we need no assumption about the structure of the Latin square. The proof is quite complicated and I have no time to describe it here.