Equitable partitions of Latin square graphs

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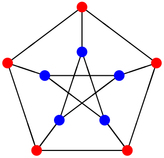
Equitable partitions

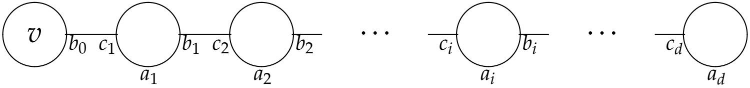
We have a graph Γ on the vertex set Ω. We assume that Γ is connected and is regular with valency k.
A partition $\{\Delta_1, \ldots, \Delta_r\}$ of Ω is equitable if there is a matrix $M = (m_{ij})$ such that a vertex in Δ_i has exactly m_{ij} neighbours in Δ_j.



In the Petersen graph, the partition of the vertices into inner and outer 5-cycle is equitable, since a vertex in one has a unique neighbour in the other.

Examples

- The orbits of a group of automorphisms of Γ. For two vertices in the same part are equivalent under an automorphism fixing all the parts.
- The distance partition with respect to any vertex v (the partition into the sets of vertices at distance $0, 1, 2, \ldots$ from v) is equitable if and only if the graph is distance-regular. The condition is equivalent to the usual definition.



In finite geometry

Many examples in finite geometry can be thought of as equitable partitions. Among these are

- sets of disjoint Steiner systems $S(t, t+1, n)$
- ovoids
- spreads
- hemisystems
- Cameron–Liebler line classes
- Many others!

The spectrum

Let Γ have adjacency matrix A. Then A is a real symmetric matrix, and so is diagonalisable by an orthogonal matrix; we refer to the spectrum of this matrix as the spectrum of Γ.

Let Δ be an equitable partition with matrix M. If $\mathbf{v}_i$ is the characteristic function of Δ_i, then

$$\mathbf{v}_j A = \sum \mathbf{v}_i m_{ij}.$$

So the subspace spanned by the vectors $\mathbf{v}_i$ is invariant under A, and the restriction of A to this space is just given by the matrix M of the equitable partition.

It follows that the spectrum of M (which we will refer to as the spectrum of the equitable partition) is contained in the spectrum of the graph Γ.

In the case of the distance partition of a distance-regular graph, the spectrum of M has all the eigenvalues of A, each with multiplicity 1.

M always has eigenvalue k, the principal eigenvalue, since its row sums are equal to k.
Let μ be an eigenvalue of Γ different from k. We say that the partition is μ-equitable if all non-principal eigenvalues of M are equal to μ. This means that the vectors $\mathbf{v}_i$ all lie in the sum of the k- and μ-eigenspaces of A.
Any two-part equitable partition is μ-equitable for some μ. But we will see that, for partitions with more than two parts, μ-equitable partitions are easier to deal with than arbitrary equitable partitions.

Perfect sets

A subset S of Ω is perfect if the partition $\{S, \Omega \setminus S\}$ is equitable; it is μ-perfect if the partition is μ-equitable.

Now easy linear algebra shows that a partition Δ is μ-equitable if and only if all but at most one part of the partition is μ-perfect.

In particular, to find all μ-equitable partitions, it suffices to find all the *minimal* μ-perfect sets.

Latin square graphs

A Latin square of order n is an $n \times n$ array with entries from an alphabet of n letters, such that each letter occurs once in each row and once in each column.
Given a Latin square L, we define the corresponding Latin square graph $\Gamma(L)$ to have as vertices the n^2 cells of the array L, two vertices joined if they lie in the same row or the same column or contain the same letter.

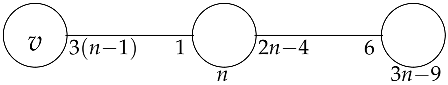
- There are n^2 vertices.
- The valency is $3(n-1)$: any cell is in the same row as $n-1$ others, in the same column as $n-1$ others, and contains the same letter as $n-1$ others.
- If two cells are joined, they have $n-2+1+1=n$ common neighbours.
- If two cells are not joined, they have $2+2+2=6$ common neighbours.

The spectrum

Thus the graph is strongly regular; its adjacency matrix A satisfies

$$A^2 = 3(n-1) + nA + 6(J - I - A),$$

where J is the all-1 matrix.



The eigenvalues of the adjacency matrix are $3(n-1)$ (the principal eigenvalue, with multiplicity 1); $n-3$ (with multiplicity $3(n-1)$), and -3 (with multiplicity $(n-1)(n-2)$).

First examples

Let S be the set of n cells in a row. Then $\{S, \Omega \setminus S\}$ is equitable, with matrix

$$\begin{pmatrix} n-1 & 2(n-1) \\ 2 & 3n-5 \end{pmatrix}.$$

Since the row sum is $3(n-1)$ and the trace is $4n-6$, the other eigenvalue is $n-3$. So S is $(n-3)$-perfect.
Of course, the same applies to any column or letter.

What G and G did

At the International Workshop on Bannai–Ito Theory in Hangzhou, Sergey Goryainov talked about a result he had proved with his supervisor Alexander Gavrilyuk. Although phrased in terms of bilinear forms, it amounted to a complete determination of the $(n-3)$-equitable partitions (or, equivalently, the minimal $(n-3)$-perfect sets) in a particular type of Latin square graph: the Cayley table of an elementary abelian 2-group.

The result is that these are rows, columns, letters, or one more type: subsquares of order $n/2$ corresponding to subgroups of index 2 in the group. See next slide.

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

RAB and PJC wondered whether this could be generalised. Certainly any Latin square of order n with a subsquare of order $n/2$ will give an example. Is that all?

Inflation

Take a Latin square L_0 of order s. Replace each occurrence of letter i by a Latin square of order t in alphabet A_i, where the alphabets for different letters are pairwise disjoint; this gives a Latin square L of order $n = st$. Moreover, given an $(s-3)$-perfect set S_0 in L_0, the corresponding cells in L form an $(n-3)$-perfect set.
For example, inflating the Latin square

0	1
1	0

using the same Latin square in each place (the Cayley table of an elementary abelian 2-group) gives the example of Gavrilyuk and Goryainov.

Corner sets

A corner set in the Cayley table of a cyclic group has shape

0	**1**	**2**	**3**	4
1	**2**	**3**	4	0
2	**3**	4	0	1
3	4	0	1	2
4	0	1	2	3

Some analysis shows that it is an $(n-3)$-perfect set.
A corner set in the order-2 Latin square is a single entry. So this extends even further the earlier examples.

The theorem

Theorem

Let S be a minimal $(n-3)$-perfect set in the graph of a Latin square of order n. Then S is a row, a column, a letter, or an inflation of a corner set.

Let S be $(n-3)$-perfect. We can assume that it contains no row, column, or letter. A slice of S is its intersection with a row, column or letter.

Choose a slice of maximum size, which (without loss) is the intersection of S with a row. There is a set of s columns meeting this row in a cell not in S. Then show that there are s rows defining slices of the same size not meeting this set of columns; these s rows and s columns form a subsquare; and the s columns are all disjoint from S.

This describes the "top right" of the given square. We use induction to work our way down and to the left, and end up showing that S is an inflation of a corner set.

−3-perfect sets

The other non-principal eigenvalue of a Latin square graph is −3. Can we say anything about −3-perfect sets? Such a set S has the property that it meets any row, column or letter in a constant number s of cells, and its cardinality is sn.
In particular, with $s = 1$, such set is a transversal, a set containing one cell from each row, column or letter.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

But not every −3-perfect set can be decomposed into transversals.

Ryser's conjecture

One of the oldest conjectures about Latin squares is Ryser's conjecture, asserting that any Latin square of odd order has a transversal. Many squares of even order do too, but some do not (for example, the Cayley table of the cyclic group). The conjecture is still open despite a lot of work, so characterising such sets is unlikely to be achieved soon!

−3-equitable partitions

A special case of a −3-equitable partition would be a partition of the elements of the Latin square into transversals.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

This is equivalent to the existence of an orthogonal mate to the given square:

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

Mixed equitable partitions

If we consider partitions where both non-principal eigenvalues occur, there is much more freedom, and probably no hope of a classification. To mention just two examples:

- The distance partition of the graph with respect to any cell. (The Latin square graph is distance-regular, so the non-principal eigenvalues each occur with multiplicity 1.)
- The $t \times t$ subsquares associated with a t-fold inflation of a Latin square of order s. (Both non-principal eigenvalues occur if and only if $s > 2$.)

And there are no doubt many more …

Late news

This morning (27 April), the paper was accepted for publication (after minor revision), just five months after we started the project.

See you at next year's Scottish Combinatorics Meeting …