# The random graph

Peter J. Cameron
University of St Andrews

Mathematics Colloquium
Worcester Polytechnic Institute
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# Example: the Petersen graph

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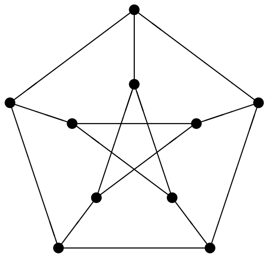

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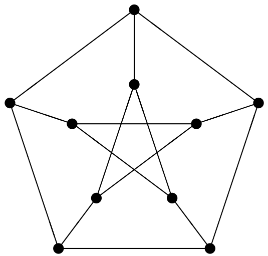

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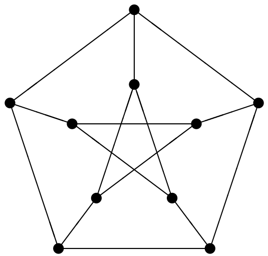

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I am going to tell you about the most famous *infinite* graph …

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I will tell you some of its story.

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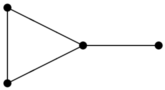

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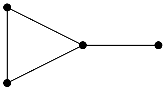

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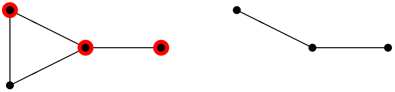

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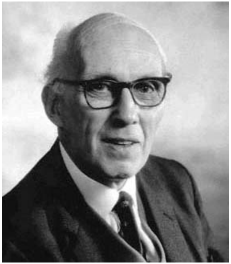
# Rado's universal graph



In 1964, Richard Rado published a construction of a countable graph which was universal. This means that every finite or countable graph occurs as an induced subgraph of Rado's graph.

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*Does $R$ have any non-trivial symmetry? And why is this very special graph the most famous infinite graph?*

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### Theorem

*There is a countable graph $R$ with the following property: if a random graph $X$ on a fixed countable vertex set is chosen by selecting edges independently at random with probability $\frac{1}{2}$, then the probability that $X$ is isomorphic to $R$ is equal to* 1.

# The proof

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I will show you the proof.

I claim that one of the distinguishing features of mathematics is that you can be convinced of such an outrageous claim by some simple reasoning. I do not believe this could happen in any other subject.

# Property (∗)

The proof depends on the following property, which a graph may or may not possess:

(∗) Given two finite disjoint sets $U$ and $V$ of vertices, there is a vertex $z$ which is joined to every vertex in $U$ and to no vertex in $V$.

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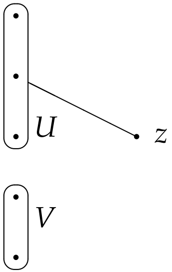


The point $z$ is called a witness for the sets $U$ and $V$.

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Then you will be convinced!

# Proof of Fact 1

We use from measure theory the fact that a countable union of null sets is null. We are trying to show that a countable graph fails $(*)$ with probability $0$; since there are only countably many choices for the (finite disjoint) sets $U$ and $V$, it suffices to show that for a fixed choice of $U$ and $V$ the probability that no witness $z$ exists is $0$.

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Since all edges are independent, the probability that none of $z_1, z_2, \ldots, z_N$ is the required witness is $\left(1 - \frac{1}{2^n}\right)^N$, which tends to 0 as $N \to \infty$.

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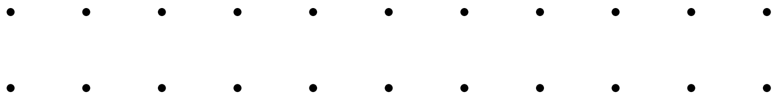
So the event that no witness exists has probability 0, as required.

# Proof of Fact 2

We use a method known to logicians as "back and forth". Suppose that $\Gamma_1$ and $\Gamma_2$ are countable graphs satisfying $(*)$: enumerate their vertex sets as $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$. We build an isomorphism $\phi$ between them in stages.



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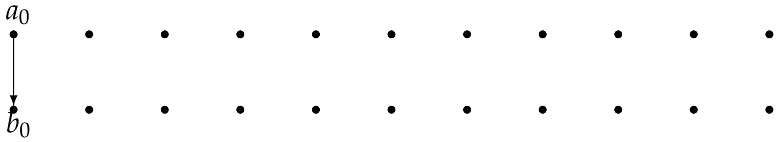


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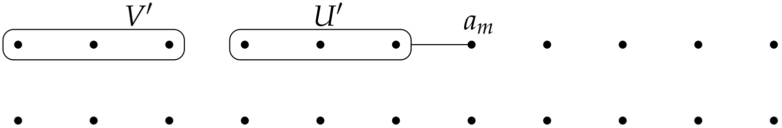


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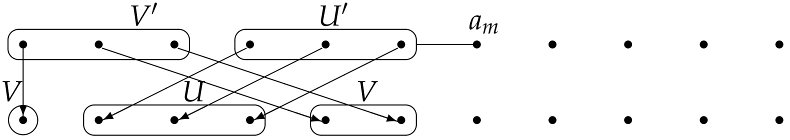


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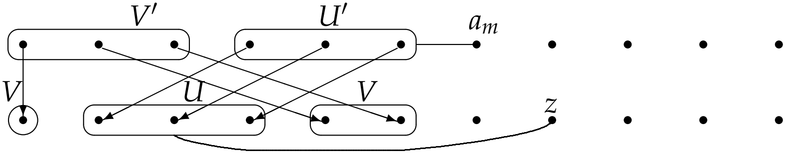


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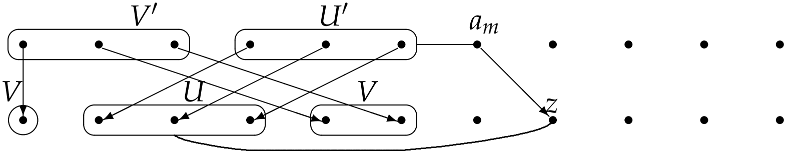


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The back-and-forth method is often credited to Georg Cantor, but it seems that he never used it, and it was invented later by E. V. Huntington.

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To see this, take $\Gamma_1 = \Gamma_2 = R$, and start the back-and-forth machine from the given finite isomorphism.

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I mentioned the problem of finding a non-trivial symmetry of this graph. There seems to be no simple formula for one! But we know it must exist. Indeed, there is a primitive recursive automorphism.

# Randomness and symmetry

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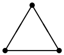
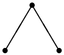
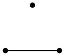
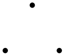
| Graph |  |  |  |  |
|---|---|---|---|---|
| Symmetries | 6 | 2 | 2 | 6 |
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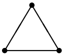
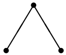
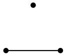
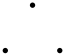
| Graph |  |  |  |  |
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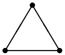
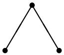
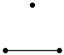
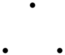
In fact, the probability of any non-trivial symmetry tends rapidly to 0 as the number of vertices increases.

# Randomness and symmetry

The fact that the random graph is highly symmetric is surprising, for several reasons.
First, for finite graphs, the more symmetric a graph, the smaller its probability of occurrence:

| Graph |  |  |  |  |
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In fact, the probability of any non-trivial symmetry tends rapidly to 0 as the number of vertices increases.
Indeed, the theorem of Erdős and Rényi was a short appendix to a long paper showing that most *finite* graphs are "as far from symmetry" as possible.

# A number-theoretic construction

Since the prime numbers are "random", we should be able to use them to construct the random graph. Here's how.

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Recall that, if $p$ is an odd prime not dividing $a$, then $a$ is a quadratic residue (mod $p$) if the congruence $a \equiv x^2 \pmod{p}$ has a solution, and a quadratic non-residue otherwise. A special case of the law of quadratic reciprocity, due to Gauss, asserts that if the primes $p$ and $q$ are congruent to 1 (mod 4), then $p$ is a quadratic residue (mod $q$) if and only if $q$ is a quadratic residue (mod $p$).

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So we can construct a graph whose vertices are all the prime numbers congruent to 1 (mod 4), with $p$ and $q$ joined if and only if $p$ is a quadratic residue (mod $q$): the law of quadratic reciprocity guarantees that the edges are undirected.

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To show this we have to verify condition ($*$). So let $U$ and $V$ be finite disjoint sets of primes congruent to 1 (mod 4). For each $u_i \in U$ let $a_i$ be a fixed quadratic residue (mod $u_i$); for each $v_j \in V$, let $b_j$ be a fixed quadratic non-residue mod $v_j$.

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By the Chinese Remainder Theorem, the simultaneous congruences

- $z \equiv a_i \pmod{u_i}$ for all $u_i \in U$,
- $z \equiv b_j \pmod{v_j}$ for all $v_j \in V$,
- $z \equiv 1 \pmod{4}$,

have a solution modulo $4 \prod u_i \prod v_j$. By Dirichlet's Theorem, this congruence class contains a prime, which is the required witness.

# The Skolem paradox

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The Skolem paradox is this: There is a theorem of set theory (for example, as axiomatised by the Zermelo–Fraenkel axioms) which asserts the existence of uncountable sets. Assuming that ZF is consistent (as we all believe!), how can this theory have a countable model?

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My point here is not to resolve this paradox, but to use it constructively.

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This graph turns out to be the random graph!

Indeed, the precise form of the axioms is not so important. We need a few basic axioms (Empty Set, Pairing, Union) and, crucially, the Axiom of Foundation, and that is all. It does not matter, for example, whether or not the Axiom of Choice holds.

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There is a simple description of a model of set theory in which the negation of the axiom of infinity holds (called hereditarily finite set theory). We represent sets by natural numbers. We encode a finite set $\{a_1, \ldots, a_r\}$ of natural numbers by the natural number $2^{a_1} + \cdots + 2^{a_r}$. (So, for example, 0 encodes the empty set.)

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When we apply the construction of "symmetrising the membership relation" to this model, we obtain Rado's description of his graph!

# Group-theoretic properties

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- As a consequence, any graph $\Gamma$ on fewer than $2^{\aleph_0}$ vertices satisfying $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(R)$ is isomorphic to $R$.
- All cycle structures of automorphisms of $R$ are known.
- $R$ is a Cayley graph for a wide class of countable groups, including all countable abelian groups of infinite exponent. For these groups, a "random Cayley graph" is isomorphic to $R$ with probability 1.

# Rolling back the years, 1

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In fact, fifteen years earlier, Roland Fraïssé had asked the question: which homogeneous relational structures exist? Fraïssé defined the age of a relational structure $M$ to be the class $\mathrm{Age}(M)$ of all finite structures embeddable in $M$ (as induced substructure). In terms of this notion, he gave a necessary and sufficient condition.

# Fraïssé classes and Fraïssé limits

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*If these conditions hold, then $M$ is unique up to isomorphism.*

A class $\mathcal{C}$ satisfying these conditions is a Fraïssé class, and the countable homogeneous structure $M$ is its Fraïssé limit.

# The amalgamation property

The amalgamation property says that two structures $B_1, B_2$ in the class $\mathcal{C}$ which have substructures isomorphic to $A$ can be "glued together" along $A$ inside a structure $C \in \mathcal{C}$:

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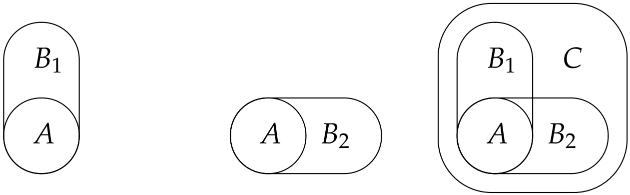

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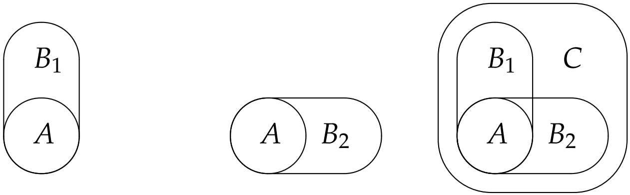


Note that the intersection of $B_1$ and $B_2$ may be larger than $A$.

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Each of the following classes is a Fraïssé class; the proofs are exercises. Thus the corresponding universal homogeneous Fraïssé limits exist.

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There are *many* others!

# Rolling back further

A quarter of a century earlier, these ideas had already been used by the Soviet mathematician P. S. Urysohn. He visited western Europe with Aleksandrov, talked to Hilbert, Hausdorff and Brouwer, and was drowned while swimming in the sea at Batz-sur-Mer in south-west France at the age of 26 in 1924.

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# Urysohn's theorem

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### Theorem

*There exists a homogeneous Polish space containing a copy of every finite metric space, and it is unique up to isometry.*

This unique metric space is known as the Urysohn space. Its study has been popularised in recent years by Anatoly Vershik.

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Its completion is easily seen to be the required Polish space.

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- The class of integral metric spaces, those with all distances integers. The Fraïssé limit is a kind of universal distance-transitive graph.
- The class of metric spaces with all distances 1 or 2. The Fraïssé limit is the random graph!

Let $M$ be the Fraïssé limit of the class of metric spaces with all distances 1 or 2; form a graph by joining two points if their distance is 1. Since the graph is homogeneous, if $v$ and $w$ are two vertices at distance 2, there is a vertex at distance 1 from both. Thus the distance in $M$ coincides with the graph distance in this graph. The graph is universal and homogeneous, and so is $R$.