Four precious jewels
2. The rational numbers

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The rationals as ordered set

By contrast to the other three examples, the history of the rational numbers stretches back for millennia.
But I will not be doing arithmetic with these numbers. I am only concerned with their ordering.
As you know, unlike the natural numbers, integers, ordinal numbers, etc., the rational numbers are dense:

> *If* $a, b \in \mathbb{Q}$ *and* $a < b$*, then there exists* $c \in \mathbb{Q}$ *such that* $a < c < b$*.*

Given this property, it came as a surprise to Cantor when he discovered that the cardinality of $\mathbb{Q}$ is the same as that of $\mathbb{N}$.
Also, there is no least or greatest rational number.

Cantor's Theorem

Cantor showed that these properties are characteristic:

Theorem

A countable totally ordered set which is dense and has no least or greatest element is order-isomorphic to $\mathbb{Q}$*.*

Now the axioms for a totally ordered set, the denseness, the absence of endpoints, can all be expressed as first-order sentences. For example, "no greatest element" translates as $(\forall x)(\exists y)(x < y)$.

So Cantor's theorem can be stated more concisely:

Theorem

$(\mathbb{Q}, <)$ *is countably categorical.*

Back-and-forth

The proof of Cantor's theorem uses an argument we saw in the last lecture: back-and-forth. To recall: Suppose that we have two countable dense ordered sets A, B without endpoints, each enumerated by the natural numbers. We build an order-isomorphism ϕ between then in stages:

- At an even-numbered stage, suppose that the map has been defined on points $a_0, \ldots, a_{n-1}$, where (without loss) $a_0 < a_1 < \cdots < a_{n-1}$. Suppose that a is the first point in the enumeration of A on which ϕ has not yet been defined, and suppose that $a_i < a < a_{i+1}$ for some value of i. Choose the first point b (in the enumeration of B) in the interval $(\phi(a_i), \phi(a_{i+1}))$ (this exists since B is dense) and extend ϕ by setting $\phi(a) = b$. If, say $a < a_0$, then a suitable b exists since B has no least element.
- At an odd-numbered stage, choose the first point in the enumeration of B not in the image of ϕ, and similarly choose a suitable preimage.

Universality, homogeneity

Just as we saw for the random graph, the argument can be modified to show

- $(\mathbb{Q}, <)$ is universal: every finite or countable total order can be embedded into it.
- $(\mathbb{Q}, <)$ is homogeneous: any order-isomorphism between finite subsets of $\mathbb{Q}$ can be extended to an order-automorphism of the whole set.

Actually, homogeneity is easy to prove directly. If $a_0 < a_1 < \cdots < a_{n-1}$ and $b_0 < b_1 < \cdots < b_{n-1}$, we can map the interval (a_i, a_{i+1}) to (b_i, b_{i+1}) by a linear map; so a piecewise-linear map carries the first n-tuple to the second.

Going forth alone

Actually this was not how Cantor proved his theorem. He defined his map in the forward direction only. It seems that back-and-forth was invented about ten years later by Huntington, in his book on Cantor's work.

Why does this work? It clearly defines a one-to-one and order-preserving map ϕ from A to B. Suppose that a point b is not in the image of ϕ. At some point in the process, b will be the point with smallest index in the interval (b_i, b_{i+1}) containing it; then at some later point in the process, it will be the image of a point in the corresponding interval in A.

Going forth alone fails in most countable structures (the random graph, the rationals partitioned into two dense subsets, etc.) It is not known precisely for which structures it always succeeds in building an isomorphism.

Set-transitivity

A permutation group G on a set Ω is said to be *k-set transitive* if any set of size k can be mapped to any other by an element of G. (This condition is often called "k-homogeneous", but here the risk of confusion is too great.) We say that G is *k-transitive* if we can map any ordered k-tuple of distinct elements to any other by an element of G.

Clearly a k-transitive group is k-set transitive. But $\mathrm{Aut}(\mathbb{Q}, <)$ is obviously k-set transitive for every natural number k, but not even 2-transitive.

What other groups have this property?

Examples

We can enlarge $\mathrm{Aut}(\mathbb{Q}, <)$ to a 2-transitive group by allowing permutations which reverse the order, as well as those that preserve it. These permutations preserve a ternary betweenness relation on $\mathbb{Q}$; so it is not 3-transitive.

Another example is obtained by bending the line to form a circle, and taking permutations preserving the ternary circular order. This group is also 2-transitive but not 3-transitive.

Finally we can combine the approaches by preserving or reversing the circular order. Such permutations preserve the quaternary separation relation, and so are 3-transitive but not 4-transitive.

Theorem

A permutation group which is k-set transitive for all k but fails to be k-transitive for some k preserves a linear order, betweenness relation, circular order, or separation relation; so it is not 4-transitive.

This was the first theorem I proved about infinite permutation groups.

Some topology

Let Ω be a countable set; for example, $\Omega = \mathbb{N}$.
There is a natural topology on the symmetric group $\mathrm{Sym}(\Omega)$, the topology of pointwise convergence: two permutations are close together if they agree on long initial subsequences of Ω.
I state without proof a couple of properties of this topology.

- The open subgroups of $\mathrm{Sym}(\Omega)$ are those containing the pointwise stabilisers of finite sets.
- The following properties of a subgroup G of $\mathrm{Sym}(\Omega)$ are equivalent:
  - G is closed in $\mathrm{Sym}(\Omega)$;
  - G is the automorphism group of a first-order structure on Ω;
  - G is the automorphism group of a homogeneous relational structure on Ω.

Given a permutation group G, there is a canonical relational structure for G, whose relations are the G-orbits on Ω^n for all n.
This structure is always homogeneous, and G is its full automorphism group if and only if G is closed.

Reducts

A reduct of a first-order structure is another first-order structure (on the same domain) whose relations, functions, etc. can be defined in terms of those of the first structure by quantifier-free formulae.

So the betweenness relation is a reduct of the linear order, defined by

$$\beta(a, b, c) \Leftrightarrow (a < b < c) \vee (c < b < a).$$

For countably categorial structures, we can define this in terms of the automorphism groups:

> B is a reduct of A if $\mathrm{Aut}(A)$ is a subgroup of $\mathrm{Aut}(B)$.

Since a group of countable degree is the automorphism group of a first-order structure if and only if it is closed (in the topology of pointwise convergence), we can say: reducts of a countably categorical structure A correspond to closed subgroups of $\mathrm{Sym}(A)$ containing $\mathrm{Aut}(A)$.

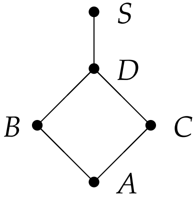
Reducts of $(\mathbb{Q}, <)$

So the theorem about set-transitive groups can be interpreted as a theorem about reducts:

Theorem

The closed subgroups of $\mathrm{Sym}(\mathbb{Q})$ *containing* $\mathrm{Aut}(\mathbb{Q}, <)$ *are* $\mathrm{Aut}(\mathbb{Q}, <)$*, the automorphism groups of the betweenness relation, the circular order, and the separation relation derived from the order on* $\mathbb{Q}$*, and the symmetric group* $\mathrm{Sym}(\mathbb{Q})$ *(preserving no structure at all).*

For brevity we call these groups A, B, C, D and S. The lattice of reducts looks like this:



Reducts of the random graph

Curiously, Simon Thomas found exactly the same lattice of reducts of the random graph!

Theorem

There are five reducts of the random graph R: $\mathrm{Aut}(R)$, the group of automorphisms and anti-automorphisms, the group of switching-automorphisms, the group of switching-automorphisms and switching-anti-automorphisms, and the symmetric group $\mathrm{Sym}(R)$.

Thomas conjectured that a countably categorical structure has only finitely many reducts. The reducts of many such structures have been explicitly calculated, and no counterexamples to the conjecture have yet been found.

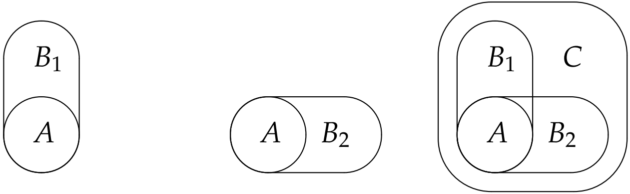
Fraïssé's Theorem

We now turn to the theorem of Fraïssé, which guarantees the existence of structures like $(\mathbb{Q}, <)$ or the random graph.
The theorem is concerned with relational structures, that is, first-order structures having no functions or constants. (Fraïssé calls these simply relations.) The age of a relational structure M is the class of all finite structures which can be embedded in M (as induced substructures). Fraïssé's theorem is a necessary and sufficient condition on a class $\mathcal{C}$ of finite structures for it to be the age of a countable homogeneous relational strucure.
Three conditions which such a class must clearly satisfy are:

- $\mathcal{C}$ is closed under isomorphism;
- $\mathcal{C}$ is closed under taking induced substructures;
- $\mathcal{C}$ contains only countably many non-isomorphic members.

The amalgamation property

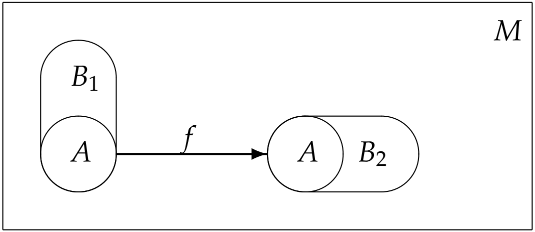
The amalgamation property says that two structures B_1, B_2 in the class $\mathcal{C}$ which have substructures isomorphic to A can be "glued together" along A inside a structure $C \in \mathcal{C}$:



Note that the intersection of B_1 and B_2 may be larger than A. Also, a technical note: I allow A to be empty; this means I don't have to state the joint embedding property separately.

The necessity of amalgamation

Let us see why the amalgamation property is necessary for homogeneity.
Suppose that M is homogeneous, and let A, B_1, B_2 be structures in the age of M.



By assumption, B_1 and B_2 are embedded in M. Take an isomorphism f between the embedded copies of A in B_1 and B_2.
By assumption, it extends to an automorphism g of M. Then $g(B_1) \cup B_2$ is the required amalgam.

The theorem

Theorem

Let $\mathcal{C}$ be a class of finite structures over a fixed relational language. Then the following conditions are necessary and sufficient for $\mathcal{C}$ to be the age of a countable homogeneous relational structure:

- *$\mathcal{C}$ is closed under isomorphism;*
- *$\mathcal{C}$ is closed under taking induced substructures;*
- *$\mathcal{C}$ contains only countably many non-isomorphic members;*
- *$\mathcal{C}$ has the amalgamation property.*

Moreover, if these conditions hold, then the countable homogeneous relational structure M is unique up to isomorphism.

A class satisfying these conditions is called a Fraïssé class, and the structure M is its Fraïssé limit.

Back to Alice's Restaurant

How do we recognise a Fraïssé limit? It turns out that a generalisation of the Alice's Restaurant property we saw for the random graph does the job.

Theorem

Let $\mathcal{C}$ be a Fraïssé class, and M a countable structure over the same language. Then M is the Fraïssé limit of $\mathcal{C}$ if and only if it has the Alice's Restaurant property with respect to $\mathcal{C}$: that is, given $A, B \in \mathcal{C}$ with $A \subseteq B$ and $|B| = |A| + 1$, every embedding of A into M can be lifted to an embedding of B into M (as induced substructure).

A little thought shows that, for graphs, this is exactly what we saw in the last lecture, while for total orders, it is the condition of being dense and having no least or greatest element.

Homogeneity and categoricity

These two concepts we have met have a close connection.

Theorem

A homogeneous countable structure over a finite relational language is countably categorical.

There are two ways of seeing this.
First, we could note that, over a finite relational language, there are only finitely many n-element structures. So, if M is a countable homogeneous structure, then $\mathrm{Aut}(M)$ has only finitely many orbits on n-tuples, that is, $\mathrm{Aut}(M)$ is oligomorphic. By Engeler *et al.*, M is countably categorical.
For another proof, we observe that the Alice's Restaurant property of a homogeneous structure over a finite language can be expressed as a set of first-order sentences; a countable model of these sentences is homogeneous, and so is isomorphic to M, by the uniqueness part of Fraïssé's theorem.

Examples

Fraïssé's motivation was the structure $(\mathbb{Q}, <)$, which is the Fraïssé limit of the class of finite total orders.
A moment's thought shows that the class of finite graphs is a Fraïssé class; its Fraïssé limit is the random graph R. Fraïssé's theorem was nearly fifteen years earlier than the work of Erdős, Rényi and Rado.
Other examples of Fraïssé classes include triangle-free graphs; partially ordered sets; bipartite graphs with a specified bipartition; permutation patterns (which can be regarded as pairs of total orders); and total orders whose elements are coloured with two colours (in this case, in the Fraïssé limit, the red and blue subsets are both dense).
However, we will see in the next lecture that Fraïssé was not the first …