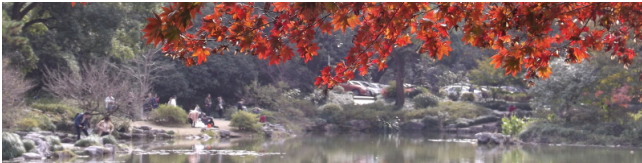


Four precious jewels
3. The Urysohn space

Peter J. Cameron
University of St Andrews

Hangzhou, August 2019



P. S. Urysohn

Pavel Urysohn, a member of the famous Lusitania group at Moscow State University led by Egorov and Lusin, produced fundamental results in topology in the early days of this subject.

You may well have met in a course on topology some of his results:

- ▶ **Urysohn's lemma:** If any two disjoint closed sets A and B in a topological space have disjoint open neighbourhoods, then for any two such sets there is a continuous real function f with $f(A) = 0, f(B) = 1$.
- ▶ **Urysohn's metrization theorem:** A normal Hausdorff space which has a countable basis of open sets is metrizable.
- ▶ He also produced one of the first dimension theories for topological spaces.

Urysohn and Alexandrov in 1924

In 1924 Urysohn and P. S. Alexandrov came to western Europe to meet mathematicians such as Hilbert, Brouwer and Hausdorff. They went for a holiday to Batz-sur-Mer in southwest France where they swam in the sea every day. On 17 August there was a storm; Alexandrov survived, Urysohn did not. He was 26.

Alexandrov and Brouwer constructed a paper from Urysohn's notes; this was published in 1925, and is the result I intend to talk about here.

I learned of this result from Anatoly Vershik at the European Congress of Mathematics in Barcelona in 2000.

Fréchet's question; Urysohn's answer

A metric space is **complete** if every Cauchy sequence converges, and is **separable** if it has a countable dense subset. Fréchet showed in 1914 that every separable metric space is isometrically embeddable in the space l_∞ of bounded real sequences with the supremum norm. But this space is not itself separable. Fréchet asked for a separable example. Urysohn showed:

Theorem

There exists a complete separable metric space \mathbb{U} with the properties

- ▶ *it is **universal**: every separable metric space can be embedded into \mathbb{U} ;*
- ▶ *it is **homogeneous**: any isometry between finite subspaces of \mathbb{U} can be extended to a self-isometry of \mathbb{U} .*

Moreover, this space is unique up to isometry.

We have seen properties like these before, for R and $(\mathbb{Q}, <)$.

Urysohn's proof foreshadowed Fraïssé's theorem a quarter of a century later, with an extra twist.

The proof

A metric space is a relational structure, with one binary relation for each distance.

First we observe that finite metric spaces have the amalgamation property. Suppose that B_1 and B_2 have a common subspace A . Without loss we can assume that $B_1 = A \cup \{b_1\}$ and $B_2 = A \cup \{b_2\}$. To form the amalgam, we have to assign a distance $d(b_1, b_2)$ without violating the triangle inequality.

We must have

$$|d(a, b_1) - d(a, b_2)| \leq d(b_1, b_2) \leq d(a', b_1) + d(a', b_2)$$

for any $a, a' \in A$. This could only fail if, without loss of generality,

$$d(a, b_1) - d(a', b_1) > d(a, b_2) + d(a', b_2),$$

which is impossible since

$$d(a, b_1) - d(a', b_1) \leq d(a, a') \leq d(a', b_2) + d(a', b_2).$$

Even though we have shown the “difficult” one of Fraïssé’s four conditions, we cannot apply his theorem, since the third condition fails. There are uncountably many non-isometric two-point metric spaces!

So instead we take \mathcal{C}_Q to be the class of finite **rational** metric spaces (those with all distances rational). Then \mathcal{C}_Q is a Fraïssé class. Let \mathbb{U}_Q be its Fraïssé limit. (This is the so-called **rational Urysohn space**.)

Now take the completion of \mathbb{U}_Q , in the usual way (Cauchy sequences modulo null sequences); this is the **Urysohn space** \mathbb{U} .

Some properties

Like the random graph, the Urysohn space has many remarkable properties.

1. It is in a rather general sense the “random” complete separable metric space. The proof of the AP for metric spaces shows that, when we are building a finite metric space one point at a time, the distances of the new point from the existing ones are constrained (by the triangle inequality) to a bounded region of Euclidean space. Now Vershik showed that if we choose the distances at random from a wide variety of probability distributions (e.g. Gaussian) on Euclidean space, and take the completion of the resulting infinite space, we obtain the Urysohn space almost surely.
2. One can use Baire category instead of measure here; the Urysohn space is “generic” in the class of Polish spaces. (I will say more about Baire category in the next lecture.)

Variations

We can vary the construction. Take \mathcal{C}_Z to be the class of finite **integral metric spaces** (with all distances integral). This is a Fraïssé class. Its Fraïssé limit has the property that the metric is the distance in the graph in which two points are joined if their distance is 1. No completion is necessary; \mathbb{U}_Z is an infinite distance-transitive graph.

Alternatively, take $\mathcal{C}_{1,2}$ to be the class of finite metric spaces with all distances 1 or 2. The Fraïssé limit again is a graph; this time, it is the **random graph** R .

Other results

The approach has been useful in other respects.

For example, various methods have been produced to construct and study subgroups of groups like the automorphism group of the random graph. These methods can be modified so as to apply to the isometry group of the rational Urysohn space, and then extended to Urysohn space by topological techniques. Here is an example, due to Vershik and me.

Theorem

Urysohn space admits an isometry of infinite order, all of whose orbits are dense in the space.

In particular, the closure of this isometry in the isometry group of \mathbb{U} (in the weak topology) is an abelian group transitive on \mathbb{U} . So \mathbb{U} has an abelian group structure (indeed, many different abelian group structures).

No complete description of the abelian group structures on \mathbb{U} is yet available.

A further example

In the same paper, we showed that the group of **bounded isometries** of \mathbb{U} (those for which the distance from a point to its image is bounded) is a proper normal subgroup of the isometry group, which is dense in the full isometry group.

Using this, and a trick due to Jacques Tits, we construct an action of the free group of countable rank as a dense subgroup of the isometry group.

Much more is surely possible, based on similar results for the random graph.

Ramsey's Theorem

For the rest of this lecture, I will discuss a connection between Ramsey theory and topological dynamics. The first three of our precious jewels all have a part to play in this story.

Theorem (Finite Ramsey theorem)

Let k, l, r be given natural numbers with $k \leq l$. Then there exists n such that, if $|X| \geq n$ and the k -element subsets of X are coloured with r different colours, then there exists an l -element subset Y of X , all of whose k -element subsets have the same colour.

Such a set Y is said to be **monochromatic**. The smallest n is the **Ramsey number** $R(k, l, r)$.

You have probably met the “party problem”: given six people at a party, either three are mutual friends or three are mutual strangers. This is the statement that $R(2, 3, 2) = 6$.

Classes of structures

The theorem is important in many fields of mathematics. For example, Paris and Harrington showed that a slight variant of it is true but unprovable in Peano arithmetic. But we will go in a different direction.

In general we replace finite sets by members of a class \mathcal{C} of relational structures. As notation, we use $\binom{B}{A}$ for the set of embedded copies of the structure A into the structure B .

For technical reasons, we often assume that our structures are rigid. In practice this is achieved by considering them to be **labelled**, that is, the point set of an n -element structure to be $\{1, 2, \dots, n\}$. Said otherwise, the language of the structures always includes a distinguished total order. This implies that any element of $\binom{B}{A}$ is the image of a unique embedding of A into B .

Ramsey classes

A class \mathcal{C} of structures as above is a **Ramsey class** if, given any two structures $A, B \in \mathcal{C}$, there is a structure $C \in \mathcal{C}$ with the property that, for every r -colouring of $\binom{C}{A}$ with r colours, there exists $B' \in \binom{C}{B}$ such that $\binom{B'}{A}$ is monochromatic.

The question is: Which classes of finite structures are Ramsey classes?

It can be shown that a the structures in a non-trivial Ramsey class must be rigid (as noted earlier).

It is convenient if Ramsey classes are required to be isomorphism-closed and **hereditary** (closed under taking substructures).

Example and non-example

The classical Ramsey theorem, together with the fact that there is a unique linear order on a finite set (up to isomorphism), shows that the class of all finite linear orders is a Ramsey class. Let \mathcal{C} be the class of finite directed graphs; let A be a single edge, and B a cyclically directed triangle.

Given any directed graph C , with a linear order on the vertices, colour an edge of C **red** if its direction agrees with the order, and **blue** otherwise. No embedded cyclic triangle is monochromatic!

Ramsey classes are Fraïssé classes

In 1989, Jarik Nešetřil showed, under mild assumptions, that an isomorphism-closed hereditary Ramsey class has the amalgamation property, and so (if it has only countably many non-isomorphic members) is a Fraïssé class.

He suggested a classification scheme, as follows. Classify the homogeneous structures; then investigate which ones have ages which are Ramsey classes.

A number of Ramsey classes have been found in this way: for example, graphs (with linear order); K_n -free graphs (with linear order); permutation patterns; metric spaces (with linear order).

Relations and functions

It is possible to extend the arguments to show that various classes of finite ordered **first-order structures** (including functions as well as relations) are Ramsey classes.

Examples include various classes of Steiner systems and resolvable designs, as well as enrichments of other graph classes such as **bowtie-free graphs**.

I refer to the nice article by Jan Hubička and Jaroslav Nešetřil in *Connections in Discrete Mathematics: A Celebration of the Work of Ron Graham*, recently published by Cambridge University Press, for further details.

Topological dynamics

Let G be a locally compact Hausdorff topological group.

A **flow** for G is a continuous action of G by homeomorphisms on a compact topological space X .

A flow is **minimal** if there is no proper subspace of X invariant under G .

A minimal flow is **universal** if there is a G -map from X to any minimal flow for G .

It can be shown that minimal flows always exist.

A topological group G is **extremely amenable** if any continuous action of G on a compact space has a fixed point. In other words, the universal minimal flow for G consists of a single fixed point.

Extreme amenability

Various examples of extremely amenable groups had been found among classical examples of topological groups. But in 2005, Kechris, Pestov and Todorćević (KPT) showed a remarkable result:

Theorem (KPT theorem)

A proper closed subgroup G of $\text{Sym}(X)$ is extremely amenable if and only if G is the automorphism group of a homogeneous structure whose age is a Ramsey class.

This theory allows information to be transferred both ways between Ramsey theory and topological dynamics. In particular, our examples above of Ramsey classes give rise to extremely amenable groups (the automorphism groups of the Fraïssé classes).

Rigidity

There are other ways of making a class of structures rigid than imposing a total order. However, with the KPT theorem, we can explain why this way of doing it must be used.

Let G be a closed subgroup of $\text{Sym}(X)$, where X is countable. Consider the set $O(X)$ of total orders on X .

It is easy to show that $O(X)$ is compact, and clearly G acts on it. So, if G is extremely amenable, then it must preserve a total order on X .

Thus in particular, if \mathcal{C} is a non-trivial Ramsey class, then the Fraïssé limit is totally ordered, so the elements of \mathcal{C} must themselves be totally ordered.

Other minimal flows

Similar methods can be used to understand the minimal flows for other closed subgroups of S_n . For example:

Theorem (Glasner–Weiss)

Let X be a countable set. Then the minimal flow for $\text{Sym}(X)$ is its action on the set $O(X)$ of all total orderings on X .

Note that $O(X)$ is the closure of the set of dense orderings without endpoints; these form a single orbit of $\text{Sym}(X)$, by Cantor's theorem.