Four precious jewels
4. The pseudo-arc

Peter J. Cameron
University of St Andrews

Hangzhou, August 2019

The unit interval

In a metric space, a continuum is a closed connected subspace.
The unit interval has a strong “homogeneity property”:

> *Any two continua in the unit interval, apart from single points, are homeomorphic.*

For, by elementary properties of $\mathbb{R}$, a closed and connected subset of $[0, 1]$ must be a closed interval.
What happens for higher-dimensional spaces like the unit square? A moment’s thought shows that the unit square contains more than one continuum (up to homeomorphism), so first we must develop a notion of “almost all”.

Baire category

This is provided by Baire category. A subset of a metric space is said to be residual if it contains a countable intersection of dense open sets.

Theorem (Baire category theorem)

In a complete metric space, a residual subset is non-empty.

Residual sets in complete metric spaces play a similar role to sets of full measure in probability spaces. For example,

- a countable intersection of residual sets is non-empty;
- a residual set has non-empty intersection with every open set.

An example

Consider the infinite binary tree, whose vertices consist of all finite strings of zeros and ones, where the two descendants of vertex v are the vertices $v0$ and $v1$ obtained by adding one extra digit to the string.
The infinite paths from the root in this tree are labelled by infinite binary strings. We can define a metric on the set of such paths by the rule that $d(s,t) = 2^{-k}$ if s and t differ first in position k.
Now it can be shown that a set S of infinite paths is

- open if and only if it is finitely determined, that is, for any $s \in S$, there is a vertex v on s such that all infinite paths containing v belong to S;
- dense if and only if it is always reachable, that is, for any vertex v, there is a path in S passing through v.

This gives a combinatorial description of the residual sets.

The random graph

Let X be a countably infinite set. Enumerate the 2-element subsets of S as $p_0, p_1, p_2, \ldots$. Now any graph on the vertex set S is represented by an infinite sequence s of zeros and ones (with $s_i = 1$ if and only if p_i is an edge of the graph).

Recall the *Alice's restaurant property* AR: for any finite disjoint sets U and V, there is a vertex z joined to everything in U and nothing in V.

For fixed U and V, this property is *finitely determined* (since its truth is determined by the status of pairs $\{x, z\}$ for $x \in U \cup V$) and *always reachable* (since after any finite number of choices, we can find a vertex z about which no decisions have been made, and add edges and non-edges so that z is a witness for $U \cup V$).

So the random graph is residual. As residual sets are non-empty, this shows its existence. This topological (non-constructive) existence proof complements the measure-theoretic proof by Erdős and Rényi.

Hausdorff metric

To apply this to our current question, we need to make the compact subsets of the unit square (or, indeed, any metric space M) into a metric space. We use the Hausdorff metric: two sets are close if every point of one is close to a point of the other. Formally, for subsets X and Y, we define

$$d_H(X, Y) = \max\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\}.$$

This has many nice properties:

- the restriction of d_H to the set $F(M)$ of all compact subsets of M is a metric;
- the induced topology on $F(M)$ depends only on the topology of M, not on the metric d;
- if M is complete, then so is $F(M)$.

The pseudo-arc

In 1920, Knaster and Kuratowski asked whether any continuum in $\mathbb{R}^2$ which is "homogeneous" (admits a transitive homeomorphism group) is necessarily a Jordan curve.

In 1922, Knaster constructed a continuum in the plane which is hereditarily indecomposable (that is, no subcontinuum is the union of two disjoint continua).

In 1948, Moise showed that Knaster's continuum is homeomorphic to any of its subcontinua. He named this structure the pseudo-arc. (As we have seen, the unit interval, and so any arc, has this property.)

In 1951 Bing showed that, up to homeomorphism, there is a unique space with the above properties. Bing also showed that the pseudo-arc is homogeneous. Furthermore, it is residual in the space of compact subspaces of the unit square, the unit hypercube in any number of dimensions, or the Hilbert cube.

Another view of the pseudo-arc

The pseudo-arc is clearly a mathematical jewel. But what does it have to do with the theme of these lectures?

In 2006, Irwin and Solecki gave a new construction of the pseudo-arc, using a result related to Fraïssé's Theorem. This was used by Solecki and Tsankov in 2015 to give a new proof of Bing's result that the pseudo-arc admits a transitive group of homeomorphisms.

These ideas have been recently developed in several places, and I hope that they will be useful in much wider contexts.

Cantor space

The famous Cantor set is given by the middle third construction, starting with the unit interval, and successively removing the middle third from each interval.

We can represent an interval after the kth step as a binary string of length k, where 0 means 'take the left-hand interval' and 1 means 'take the right-hand interval'.
Alternatively, the intervals at the nth step are the paths in the complete binary tree of height n.
So points in the Cantor space are represented by infinite binary sequences, or paths in the binary tree. The induced topology as subspace of the unit interval coincides with the topology from the metric defined earlier.

Cantor space and unit interval

The Cantor space is a subspace of $[0,1]$. Both have cardinality $2^{\aleph_0}$, but their topologies are very different.

However, we have a map between them which is almost a bijection: regard an infinite binary sequence $(e_0, e_1, e_2, \ldots)$ as a 'binary decimal' $0.e_1e_2e_3\cdots$.

The map just defined is bijective except on a countable set consisting of sequences which are constant (0 or 1) after some point:

$$0.e_1e_2\cdots e_k0111\cdots = 0.e_1e_2\cdots e_k1000\cdots .$$

So the unit interval can be obtained as a quotient of Cantor space, by identifying pairs of the above form.

A combinatorial version

We can produce the above identification another way.
We have a natural ordering on the points of the nth approximation: lexicographic order on the binary sequences, or depth-first search order on the paths in the tree. Let us call two points neighbours if they are equal or adjacent in this order.
What relation is induced on Cantor space by the neighbour relation? In other words, when do two infinite binary sequences have the property that their truncations to length n are neighbours for all n? It is easy to see that this holds if and only if either the infinite sequences are equal, or they have the form

$$e_1 e_2 \cdots e_k 0111 \cdots \text{ and } e_1 e_2 \cdots e_k 1000 \cdots ,$$

in other words, if and only if they represent the same point of the unit interval.

So the neighbour relation on Cantor space is an equivalence relation with equivalence classes of sizes 1 and 2, and the quotient is the unit interval.

The points of the unit interval which come from classes of size 2 are the 2-adic rationals. But notice that, after the identification, we can no longer recognise these points; the group of self-homeomorphisms of $[0, 1]$ is transitive on the set of all interior points.

Projective limits

The amalgamation property of a class $\mathcal{C}$ can be regarded as a commutative square or pushout in the category whose objects are the elements of $\mathcal{C}$ and and whose morphisms are injective homomorphisms.

$$\begin{array}{ccc}
B_1 & \xrightarrow{g_1} & C \\
{\scriptstyle f_1}\big\uparrow & & \big\uparrow{\scriptstyle g_1} \\
A & \xrightarrow[f_2]{} & B_2
\end{array}$$

A category theorist would certainly ask, "What happens if you dualise by turning the arrows around?" We would replace embeddings (injective homomorphisms) by projections (surjective homomorphisms).

Inverse limits

Let $\{0,1\}^n$ denote the set of words of length n over the alphabet $\{0,1\}$.
If $i < j$, we have a projection $\pi_{j,i}$ from $\{0,1\}^j$ to $\{0,1\}^i$, by simply deleting the last $j - i$ entries. If $i < j < k$, then $\pi_{k,j} \circ \pi_{j,i} = \pi_{k,i}$.
This is called an inverse system. Its inverse limit is an object L with projections $\phi_i : L \mapsto \{0,1\}^i$ for all i such that, for $i < j$, $\phi_j \circ \pi_{j,i} = \phi_i$.
Clearly the inverse limit is the set of all binary sequences, and the map ϕ_i removes all but the first i entries in a sequence.
Thus, Cantor space is the inverse limit of the system of finite binary words and projections $\pi_{j,i}$ defined above.

Projective amalgamation property

We would be looking at the following diagram, where morphisms are surjective homomorphisms:

$$\begin{array}{ccc} B_1 & \xleftarrow{f_1} & A \\ {\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ C & \xleftarrow[g_2]{} & B_2 \end{array}$$

If, for any $A, B_1, B_2 \in \mathcal{C}$ and epimorphisms $f_i : A \to B_i$ for $i = 1, 2$, there exists $\mathrm{C} \in \mathcal{C}$ and epimorphisms $g_i : B_i \to \mathrm{C}$ such that the diagram commutes, then we say that $\mathcal{C}$ has the projective amalgamation property.

Projective Fraïssé classes

We say that a class $\mathcal{C}$ of finite structures is a projective Fraïssé class, or dual Fraïssé class, if it has the following properties:

- $\mathcal{C}$ is closed under isomorphism.
- $\mathcal{C}$ is closed under taking epimorphic images.
- $\mathcal{C}$ contains only countably many non-isomorphic members.
- For any $B_1, B_2 \in \mathcal{C}$, there exists $C \in C$ which projects onto both of them.
- $\mathcal{C}$ has the projective amalgamation property.

The fourth condition is the dual of the joint embedding property. We managed to avoid the JEP in Fraïssé's Theorem by assuming that the empty set embeds into any structure. But no non-empty structure projects onto the empty set, so we need to state it separately here.

The projective Fraïssé theorem

Irwin and Solecki proved the following.

Theorem

Let $\mathcal{C}$ be a projective Fraïssé class. Then there is a structure M with the properties that $\mathcal{C}$ is the set of all finite projections (epimorphic images) of M, and if $\pi_1, \pi_2 : M \to A$ are projections to $A \in \mathcal{C}$, then there is an automorphism g of M such that $g \circ \pi_1 = \pi_2$. The structure M is unique up to isomorphism.

The structure M is the projective Fraïssé limit of the class $\mathcal{C}$.

Back to the pseudo-arc

A loopy path is a graph which consists of a finite path with a loop at each vertex.
Irwin and Solecki showed that the class of loopy paths is a projective Fraïssé class.
The graph structure of its projective Fraïssé limit M is rather straightforward: it consists of uncountably many components which consist of a single vertex with a loop, and countably many which consist of a loopy path of length 2.
They showed that, if we use the profinite topology on M (induced from the product topology of the discrete spaces on members of $\mathcal{C}$, and then factor out the equivalence relation corresponding to the connected components, then the resulting space is the pseudo-arc.

Homogeneity

As noted, in this area the term "homogeneous" just means that the group of self-homeomorphisms acts transitively on the points.
Using this approach, Solecki and Tsankov were able to give a new proof of Bing's result that the pseudo-arc is homogeneous.
As with Cantor space, once we factor out the equivalence relation, the points which come from equivalence classes of size 2 are no longer distinguished.
Are there any stronger transitivity conditions? Note that the "projcetive homogeneity" in the Irwin–Solecki theorem does not easily translate into the action on points of the space.
The homeomorphism group of the pseudo-arc is surely an interesting group which deserves further study!

Other applications

Meanwhile, Dolinka and Mašulović were also developing the idea of injective and projective Fraïssé limits with different applications in mind.
One of their aims is an application of these ideas to universality properties of the endomorphism monoids of (injective or projective) Fraïssé limits.
For more details, see the slides of Dolinka and Solecki at
`http://www.maths.dur.ac.uk/lms/102/talks.html`
(There are many other interesting sets of slides on this site!)

The end . . .

That is the end of my story, though hopefully the beginning of new research and collaboration.

for your attention!