Growth rates for oligomorphic groups

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Introduction

Let G be a permutation group on a set Ω, which in this talk is almost always infinite.

We say that G is oligomorphic if the number $f_n(G)$ of orbits of G on the set of n-element subsets of Ω is finite for all n.

These permutation groups have many connections with logic (countably categorical first-order theories), theory of relations (homogeneous relational structures), and of course combinatorial enumeration.

For some time, the question of interest was:

Problem

How fast does the sequence $(f_n(G))_{n\in\mathbb{N}}$ *grow?*

Very recently, the work of Justine Falque, Nicolas Thiéry, Pierre Simon, and Samuel Braunfeld has given surprisingly sharp answers to questions I raised some time ago.

Countably categorical structures

A countable structure over a first-order language is countably categorical if it is the unique countable model of its first-order theory (up to isomorphism).

In 1959, Engeler, Ryll-Nardzewski and Svenonius independently proved versions of the following theorem:

Theorem

A countable first-order structure is countably categorical if and only if its automorphism group is oligomorphic.

This is one of my favourite theorems: it says, informally, "symmetry is equivalent to axiomatisability".

We will see *many* examples of such structures in this talk.

Homogeneous structures

A relational structure is a structure over a relational first-order language (no functions or constants). Examples: Graphs, posets, hypergraphs, ...
Relational structures were studied by Roland Fraïssé. He defined the age of a relational structure to be the class of finite structures embeddable in it. Also, a relational structure M is homogeneous if any isomorphism between finite substructures of M can be extended to an automorphism of M.
One of Fraïssé's most celebrated theorems is the following:

Theorem

A class $\mathcal{C}$ is the age of a countable homogeneous relational structure M if and only if it is closed under isomorphism, closed under taking substructures, has only countably many members (up to isomorphism), and has the amalgamation property. If these conditions hold, then M is uniquely determined by $\mathcal{C}$ up to isomorphism.

Fraïssé classes and Fraïssé limits

The amalgamation property says that two structures with isomorphic substructures can be glued together along the given isomorphism inside a member of the class $\mathcal{C}$.

If the conditions of Fraïssé's Theorem hold, then $\mathcal{C}$ is called a Fraïssé class, and M is the Fraïssé limit of $\mathcal{C}$.

This is a very powerful method of constructing homogeneous structures. For example, the class of finite graphs is a Fraïssé class, and the countable random graph is its Fraïssé limit.

Note that, if there are only finitely many relation symbols in the language, then the automorphism group of a homogeneous structure is oligomorphic.

Closure

There is a natural topology on the symmetric group on an infinite set Ω: a basis for the open sets consists of the cosets of pointwise stabilisers of finite sets.

If Ω is countable, then this topology is metrisable: taking $\Omega = \mathbb{N}$, two permutations are close together if they agree on a long initial subset of $\mathbb{N}$.

A subgroup of $\mathrm{Sym}(\Omega)$ is closed in this topology if and only if it is the full automorphism group of a first-order structure. We can take this structure to be a homogeneous relational structure.

So we can restrict our attention to closed permutation groups, that is, automorphism groups.

The bottom of the hierarchy

A permutation group G is n-homogeneous if $f_n(G) = 1$; it is highly homogeneous if it is n-homogeneous for all $n \in \mathbb{N}$.
Highly homogeneous groups are obviously at the bottom of the growth rate hierarchy.

Theorem

Suppose that $(f_n(G))$ is bounded. Then G fixes a finite subset of Ω and acts on its complement as a highly homogeneous permutation group.

For example, the direct product of a highly homogeneous group and a finite group.
So first we need to describe the highly homogeneous groups.
The first class of examples we notice are groups of order-preserving permutations of a suitable totally ordered set, such as $\mathbb{Q}$ or $\mathbb{R}$.

Highly homogeneous groups

We can find another example by allowing permutations which preserve or reverse the order; this preserves a ternary betweenness relation, and so is 2-transitive but not 3-transitive. Again, we can bend the line into a circle and preserve the circular order; this group is again 2-transitive but not 3-transitive.
Combining the two constructions we obtain an example which is 3-transitive but not 4-transitive.

Theorem

If G is highly homogeneous but not highly transitive, then there is a linear or circular order preserved or reversed by all elements of G; in particular G is not 4-transitive.

It follows that there are exactly five closed highly homogeneous permutation groups of countable degree. The structures they act on are the reducts of $(\mathbb{Q}, <)$.

Direct product

If H and K are permutation groups on sets Γ and Δ, we can build an action of $G = H \times K$ on the disjoint union of Γ and Δ.

One can calculate $f_n(H \times K)$ in terms of $f_n(H)$ and $f_n(K)$. This is most easily expressed in terms of the generating functions, for these sequences.

For any permutation group G, put $\phi_G(x) = \sum_{n \geq 0} f_n(G) x^n$. Then

Proposition

$\phi_{H \times K}(x) = \phi_H(x) \phi_K(x)$.

Note that G is highly homogeneous if and only if $\phi_G(x) = (1 - x)^{-1}$. Hence, for such a G, we have $\phi_{G^r}(x) = (1 - x)^{-r}$, from which it follows that $f_n(G^r)$ is a polynomial of degree $r - 1$ in n.

Imprimitivity and wreath products

The permutation group G is imprimitive if it preserves a non-trivial equivalence relation on Ω.

If G is imprimitive, we can build two "smaller" groups:

- H, the stabiliser of an equivalence class, acting on that class;
- K, the group induced by G on the set of equivalence classes.

Conversely, given permutation groups H and K, we can build the largest imprimitive group with H as class stabiliser and K as group induced on the classes: this is the wreath product $H \operatorname{Wr} K$.

The orbit algebra

Let G be an permutation group. The orbit algebra of G is the graded algebra $A^G = \bigoplus_{n\geq 0} V_n^G$, where V_n^G is the vector space of functions from $\binom{\Omega}{n}$ (the set of n-subsets of Ω) to $\mathbb{C}$ which are fixed by G (that is, constant on G-orbits).
The multiplication is given as follows: if $f_i \in V_{n_i}^G$ for $i = 1, 2$, then $f_1 f_2$ is the function in $V_{n_1+n_2}^G$ whose value on an $(n_1 + n_2)$-set A is given by

$$(f_1 f_2)(A) = \sum_{\substack{B\subseteq A \\ |B|=n_1}} f_1(B) f_2(A \setminus B).$$

This graded algebra is commutative and associative, and contains an identity (the constant function 1 in V_0^G).

The Hilbert series

If G is oligomorphic, then the dimension of the nth homogeneous component V_n^G of A^G is $f_n(G)$. So the Hilbert series of A^G is

$$\sum_{n\geq 0} \dim(V_n^G)x^n = \phi_G(x).$$

In the case where H is highly homogeneous and K is finite, $A^{H\mathrm{Wr}K}$ is the algebra of invariants of K, an algebra whose structure is well known; in particular, it is finitely generated. The series $\phi_{H\mathrm{Wr}K}(X)$ is its Molien series.

For example, if K is the symmetric group of degree r, then its invariants are generated by the elementary symmetric polynomials in r indeterminates (Newton's Theorem), and we have

$$\phi_{H\mathrm{Wr}K}(x) = (1-x)^{-1}(1-x^2)^{-1}\cdots(1-x^r)^{-1}.$$

The theorem of Falque and Thiéry

We saw that, if H is highly homogeneous and K is finite, and $G = H \operatorname{Wr} K$, then A^G is finitely generated, so $(f_n(G))$ grows polynomially, and we know a lot about the structure of A^G. Falque and Thiéry proved a very strong generalisation of this observation, which was extended by Falque in her PhD thesis to give detailed structural information:

Theorem

Let G be an oligomorphic permutation group, and suppose that $f_n(G)$ is bounded by a polynomial in n. Then A^G is a Cohen–Macaulay algebra. In consequence, there are numbers $a > 0$ and $r \in \mathbb{N}$ such that

$$f_n(G) \sim an^r.$$

Cohen–Macaulay algebras

Briefly, an algebra is Cohen–Macaulay if it is finite-dimensional over a subalgebra generated by a finite set of algebraically independent elements.

Examples include coordinate rings of smooth algebraic varieties.

The Hilbert series of a Cohen–Macaulay algebra is a rational function with denominator of the form

$$\prod_{i=1}^{r}(1 - x^{d_i}),$$

where the d_i are degrees of the alebraically independent elements.

Imprimitive groups, again

In what follows, S denotes the symmetric group of infinite degree.
If G is imprimitive, with infinitely many infinite blocks, then $G \leq S \operatorname{Wr} S$, and so $f_n(G) \geq f_n(S \operatorname{Wr} S)$. It is easy to see that $f_n(S \operatorname{Wr} S) = p(n)$ is the classical partition function, the number of partitions of the integer n.
Hardy and Ramanujan calculated the asymptotics:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

In broad-brush terms, it grows roughly like $\exp(n^{1/2})$.
More generally, if G is a group with $f_n(G) \sim n^d$ the, in similar broad-brush terms, $f_n(G \operatorname{Wr} S)$ is about $\exp(n^{(d+1)/(d+2)})$, so slower than exponential.

Another imprimitive example

Let $G = C_2 \operatorname{Wr} A$, where A denotes the group of order-preserving permutations of $\mathbb{Q}$. The domain Ω can be regarded as being partitioned into sets of size 2, and the parts carry an order isomorphic to $\mathbb{Q}$. The points within a pair can be permuted arbitrarily, and the pairs can be permuted so as to preserve the order.

An n-set contains either or both points from each of a certain number of pairs; if just one, it doesn't matter which. So the orbit is described by an ordered sequence of 1s and 2s which sums to n. It is a classical result that these sequences are counted by the Fibonacci number F_n.

It is also well known that $F_n \sim c \cdot \alpha^n$, where α is the golden ratio $\frac{1}{2}(1 + \sqrt{5})$.

So this group, though imprimitive, exhibits exponential growth.

Just below exponential

For any oligomorphic group G, the generating function for $G \operatorname{Wr} S$ (where S is an infinite symmetric group) is given by

$$\sum_{n\geq 0} f_n(G \operatorname{Wr} S)x^n = \prod_{n\geq 1}(1-x^n)^{-f_n(G)}.$$

From this we can work out the asymptotics:

- if $f_n(G)$ is polynomial of degree d, then $f_n(G \operatorname{Wr} S)$ is fractional exponential, roughly $\exp(n^{(d+1)/(d+2)})$;
- if $f_n(G)$ is faster than polynomial, then $f_n(G \operatorname{Wr} S)$ is faster than fractional exponential;
- if $f_n(G)$ is slower than exponential, then so is $f_n(G \operatorname{Wr} S)$.

In particular, $f_n(S \operatorname{Wr} S \operatorname{Wr} S$, $f_n(S \operatorname{Wr} S \operatorname{Wr} S \operatorname{Wr} S)$, …, all grow faster than fractional exponential but slower than exponential.

Local orders

A local order, or locally transitive tournament, is a tournament (a complete graph with every edge carrying a direction) in which no cyclic triple dominates or is dominated by a vertex. The number of n-vertex local orders, up to isomorphism, is about $2^{n-1}/n$.

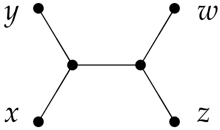
Local orders form a Fraïssé class. Its Fraïssé limit can be described as follows. Choose a countable dense subset of the set of roots of unity on the unit circle with the property that exactly one of each antipodal pair is chosen. (There is an essentially unique way to do this.) Now these points are the vertices, and we direct the edge $\{x, y\}$ from x to y if the (positive-sense) arc from x to y is shorter than the arc from y to x.

Now the automorphism group G of this local order satisfies $f_n(G) \sim 2^{n-1}/n$.

By allowing order reversals, this number is approxiately halved. These groups are primitive.

Ends of boron trees

A boron tree is a tree in which all vertices have valency 1 or 3. (Think of it as the analogue of a hydrocarbon molecule where trivalent boron replaces tetravalent carbon.)
There is a quaternary relation on the set of leaves of a boron tree, where $R(x, y \mid z, w)$ holds for distinct x, y, z, w if the paths from x to y and from z to w are disjoint (this condition holds for only one of the three partitions of the four arguments into two 2-sets).



These structures again form a Fraïssé class; and the number of them is asymptotically $cn^{-5/2}(2.483\ldots)^n$. So again we get a primitive group with exponential growth.

Permutation patterns

Let $\mathcal{C}$ be the class of finite sets carrying two (independent) total orders. Since the second is a permutation of the first, the number of n-element structures in $\mathcal{C}$ is $n!$.

$\mathcal{C}$ is a Fraïssé class; its Fraïssé limit is the generic biorder, whose automorphism group has $f_n(G) = n!$. By Stirling's formula, $n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n$. So the growth is a little faster than exponential.

This example is related to the theory of permutation patterns, a topic of interest to several people here.

Binary matrices

Let $\mathcal{C}$ be the class of structures corresponding to the entries 1 in $(0,1)$-matrices with no zero rows or columns, up to row and column permutations. We can think of the domains of $\mathcal{C}$-structures as the edge sets of bipartite graphs. (I will not be too precise about what the actual relations are.) Then the number of n-element $\mathcal{C}$-structures is quite difficult to evaluate, even asymptotically; but, with Thomas Prellberg and Dudley Stark, I found it to be roughly $(n/(\log n)^2)^n$, which grows faster than exponentially but slower than the factorial function.

As usual, $\mathcal{C}$ is a Fraïssé class. So we get an oligomorphic group with this growth rate.

Faster growth

For the automorphism group of the random graph, $f_n(G) \sim 2^{n(n-1)/2}/n!$, the number of n-vertex graphs up to isomorphism.

Using Fraïssé's Theorem, it is easy to show that there is no upper bound for the rate of growth of $(f_n(G))$.

However, if G is the automorphism group of a homogeneous relational structure over a finite relational language, then $f_n(G)$ is bounded by the exponential of a polynomial. Specifically, if the relations have arities $a_1, \ldots, a_r$ then

$$f_n(G) \leq 2^{n^{a_1}+\cdots+n^{a_r}}.$$

Some conjectures

On the basis of these and other examples, I was led to several conjectures. I will state the conjectures just for primitive groups. (All the groups from local orders onwards have been primitive.)

Let G be primitive but not highly homogeneous, and $(f_n(G))$ the corresponding sequence. Then I conjectured that

- $f_n(G)$ grows rapidly (at least exponentially);
- in particular, $f_n(G) \geq 2^n/F(n)$, for some polynomial F (possibly depending on G);
- $f_n(G)$ grows "smoothly" (see next slide).

I will conclude by reporting progress on these conjectures.

Smooth growth?

There are two ways to interpret this.
Locally, we might require relations between a few consecutive values of $f_n(G)$, along the lines of log-convexity. (Not all growth sequences are log-convex – for example, if $G = S_\infty \operatorname{Wr} S_2$ or $S_2 \operatorname{Wr} S_\infty$, then $f_n(G) = 1 + \lfloor n/2 \rfloor$ – but it seems nearly true.

Globally, we could require that certain limits exist, for example

- $\log f_n(G) / \log n$, for polynomial growth (the result of Falque and Thiéry shows that this limit exists and is a non-negative integer if it is finite);
- $\log \log f_n(G) / \log n$, for fractional exponential growth (the results of Simon and Braunfeld give information);
- $\log f_n(G)/n$, for exponential growth.

I have no general conjecture to offer.

Macpherson's first result

At the start of his DPhil, Dugald Macpherson proved that, in fact, $f_n(G) \geq n^{1/2-\epsilon}$; in other words, the growth is comparable to the partition function.

He did this by encoding partitions of n into the orbits of G on cn-sets, for some small constant c.

Subsequently, we know from the work of Simon and Braunfeld that, in fact, for any oligomorphic group, if $(f_n(G))$ grows faster than polynomially, then its growth is of comparable in this sense to the partition function.

Exponential growth

In his thesis, Dugald proved that there is a universal constant $c > 1$ such that, if G is primitive but not highly homogeneous, then in fact $f_n(g) \geq c^n / F(n)$, where F is a polynomial (depending on G). He gave $g = \sqrt[5]{2} = 1.149\ldots$.
Later, Francesca Merola improved the constant to about $1.324\ldots$, by using Dugald's argument a little more carefully. There are a number of structures a primitive group may preserve, which all give different lower bounds; the constant comes from the worst case.
Then Pierre Simon improved it to $1.576\ldots$, by observing that Merola's worst case could not in fact occur; so we jump up to the next case.
Finally, in a very recent preprint, Sam Braunfeld has improved the constant to 2. We know from the local orders that no further improvement is possible. I'll say a bit about the methods at the end.

Exponential constants

Problem

For which real numbers $c > 1$ is there a primitive oligomorphic group G for which $f_n(G)$ is "about" c^n (in the sense that the ratio is bounded by a power of n)?

We know only countably many points in this *exponential growth rate spectrum*, of which 2 and $2.483\ldots$ are the smallest.

Could it be true that there are only countably many possible values? Much more flexible constructions would be needed to refute this.

Stability

It was Macpherson who first realised that model-theoretic conditions on a countably categorical theory would have implications for the growth rate. This has been extended by Simon and Braunfeld.

A theory is κ-stable if it has at most κ complete types; it is stable if it is κ-stable for some infinite cardinal κ.

Roughly, a theory is unstable if it encodes the theory of the natural numbers.

Stability theory originated in Morley's Theorem (proving the Łoś conjecture) that a theory in a countable language categorical in one uncountable cardinality is uncountable in all.

The connection

Pierre Simon showed that, if M is a countably categorical structure whose automorphism group G is primitive, then one of the following holds:

- M is bi-definable with $(\mathbb{Q}, <)$ or one of its reducts (and hence G is highly homogeneous);
- M is stable but not ω-stable;
- $f_n(G) \geq 2^n / F(n)$ for some polynomial F.

In particular we have exponential growth in the unstable case.

Monadic stability

A theory T is monadically stable if every expansion of T by unary predicates is stable.
Sam Braunfeld proved:

Theorem
Let T be a countably categorical theory whose countable model has (oligomorphic) automorphism group G. Suppose that T is stable. Then either

- *T is monadically stable, and $(f_G(n))$ grows slower than exponentially; or*
- *T is not monadically stable, and $(f_G(n))$ grows faster than exponentially.*

There is more to it, but these ideas are at the basis of what I have reported.

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