Four precious jewels

Peter J. Cameron
University of St Andrews

Virtual Combinatorics Colloquium
Northeast Combinatorics Network
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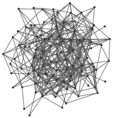
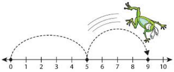

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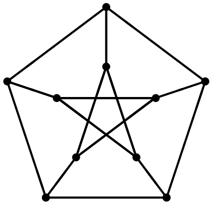
- the rational numbers, as ordered set;
- the Urysohn metric space;
- the pseudo-arc.

In each case, the object can be constructed and studied by methods of finite combinatorics (usually some variant on Fraïssé's amalgamation method); and they in turn contribute to areas of finite combinatorics such as Ramsey theory, as well as further afield in topological dynamics.

Finite random graphs

In 1963, Paul Erdős and Alfred Rényi wrote a paper on "Asymmetric graphs". They showed that, not only does a random finite graph (edges chosen independently with probability $\frac{1}{2}$) have no non-trivial automorphisms (with high probability), but it is in a certain sense at maximum distance from symmetry.

So beautiful symmetric objects like the Petersen graph are rare.



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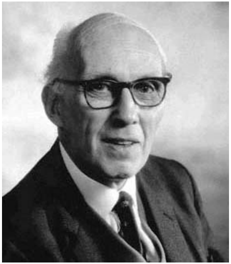
This one remarkable graph has many more properties, and arises in many different parts of mathematics, as you might expect.

Rado's graph



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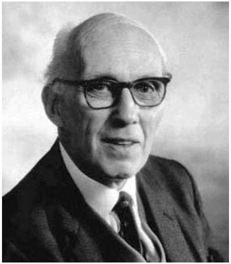

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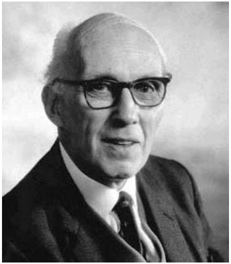

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Here is Rado's construction. The vertex set of his graph is the set of natural numbers (including 0); for $x < y$, we join x to y if the xth digit in the base 2 expansion of y is equal to 1.

There are many other constructions of this graph. For example, take the vertices to be the primes congruent to 1 (mod 4); join p to q if p is a quadratic residue mod q.

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Rado's graph turns out to be "the" Erdős–Rényi random graph!

Why does it work?

The property of the graph R used both by Erdős and Rényi and by Rado is the following, sometimes called the Alice's Restaurant property ("you can get anything you want"):

($*$) Given finite disjoint sets U and V of vertices of R, there is a vertex z joined to everything in U and to nothing in V.

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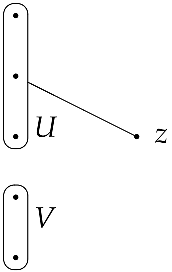

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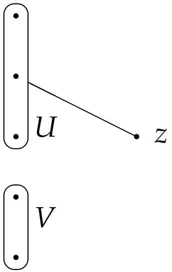

This property says that given any finite subgraph W of R, every possible extension of W to a graph with one extra vertex is realised inside R.

It is a fairly easy calculation to show that this holds with probability 1 in a countable random graph. (There are countably many choices of U and V; for each choice of U and V, z exists with probability 1; and a countable intersection of sets of measure 1 has measure 1.)

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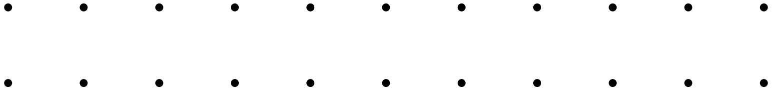
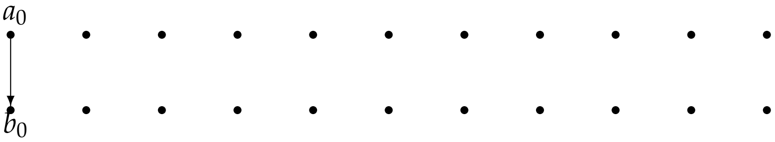
The picture on the next slide demonstrates the process.





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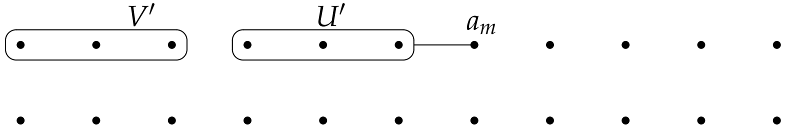

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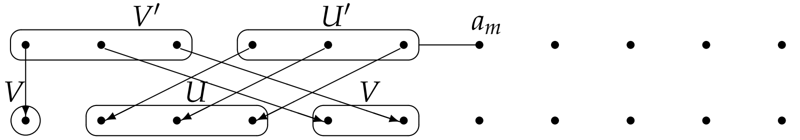

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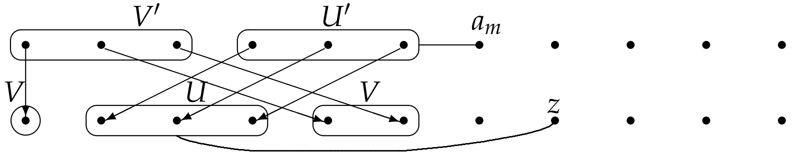

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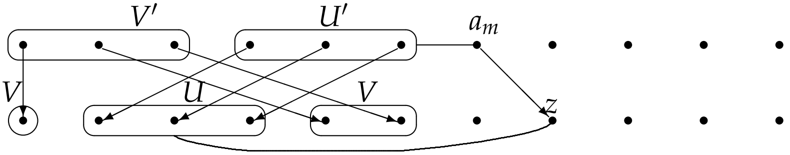

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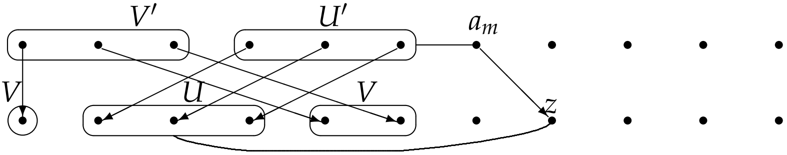

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To produce an isomorphism, we must go "back and forth": at odd-numbered stages, choose the first unused vertex in B, and use property $(*)$ in A to find a preimage for it.

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This is very easy for Rado's construction. For the number theory construction, it is a pleasant exercise in nineteenth century number theory (quadratic reciprocity and Dirichlet's theorem are needed). For the set theory construction, it is an easy deduction from the axioms (only Null Set, Pairing, Union and Foundation are required).

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This is the basis for a wide-ranging study of the automorphisms and endomorphisms of R; but that is another topic!

The rational numbers

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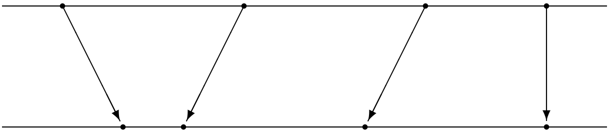

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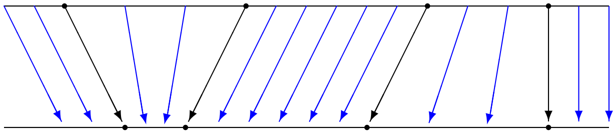

Fraïssé

In the late 1940s and early 1950s, Roland Fraïssé gave a wide generalisation of the fact that the rational numbers are homogeneous and universal for finite and countable ordered sets. The context is relational structures over a given relational language $\mathcal{L}$, that is, sets carrying relations of the arities specified by the language. (For example, for graphs or total orders we take a single binary relation.)

Maps between structures will be embeddings as induced substructures: that is, $f : A \to B$ carries each instance of a relation in A to an instance of the corresponding relation in B, and each instance of a relation in the image of A arises in this way. Remember: for graphs, we are using induced subgraphs.

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I allow the empty set as a structure, but require it to be unique (that is, there are no "nullary relations" in the language).

The age of a relational structure A is the class of all finite structures embeddable in A as induced substructures.

A class $\mathcal{A}$ of finite structures is hereditary if it is closed under taking substructures. It has the amalgamation property if two structures B_1, B_2 in the class $\mathcal{A}$ which have substructures isomorphic to A can be "glued together" along A inside a structure $C \in \mathcal{A}$:

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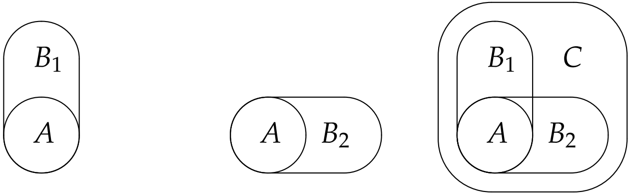

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A class $\mathcal{A}$ satisfying these conditions is called a Fraïssé class, and the countable homogeneous structure M is its Fraïssé limit.
The Alice's restaurant property of the Fraïssé limit M says that, if B and C are elements of the age of M with $B \subseteq C$ and $|C| = |B| + 1$, then any embedding of B in M can be extended to an embedding of C. Then everything works just as for the random graph.

Examples

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There are *many* others!

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Pavel Samuilovich Urysohn was a Soviet pioneer of topology. He came to western Europe with Aleksandrov, and met Brouwer and Hilbert. On holiday in the south of France, he was drowned while swimming in the sea, at the age of 26. One of his last pieces of work was later written up by Aleksandrov and Brouwer.

Urysohn's Theorem

A Polish space is a complete separable metric space: that is, one in which all Cauchy sequences converge, and there is a countable dense subset. Urysohn showed that there is a universal homogeneous Polish space: that is, every Polish space is embeddable in Urysohn's space, and any isometry between finite subsets extends to an isometry of the whole space.

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We will look at this result from a "post-Fraïssé viewpoint".

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Incidentally, if we play the same game with metric spaces with all distances integral, we obtain a homogeneous distance-transitive graph; and if we use just distances 1 and 2, we obtain the random graph!

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The aim is to generalise this from a theorem about sets to a theorem about structures such as graphs. In order to do this, it is convenient to replace "subsets" by "embeddings".

Accordingly, if A and B are structures, we denote by $\binom{B}{A}$ the set of all embeddings of A into B.

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A class $\mathcal{A}$ of finite structures is a Ramsey class if, given $A, B \in \mathcal{A}$ and $r > 1$, there is a structure $C \in \mathcal{A}$ such that, if elements of $\binom{C}{A}$ is coloured with r colours, there exists an element $f \in \binom{C}{B}$ (an embedding of B into C) such that all embeddings of A into C with image contained in the image of f have the same colour.

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It turns out that this can only hold if all the structures in $\mathcal{A}$ have trivial automorphism group. This can be achieved by considering them as labelled structures, that is, the point set is $\{1, \ldots, n\}$ for some n (i.e. is totally ordered).

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In particular, several examples of Fraïssé classes of labelled finite objects that we have seen turn out to be Ramsey classes, including graphs, triangle-free graphs, permutations, and metric spaces.

Topology of automorphism groups

There is a natural topology defined on the symmetric group on a countable set such as $\mathbb{N}$, the topology of pointwise convergence: two permutations g and h are close together they agree on a long initial segment of $\mathbb{N}$. Without going into details, here are a couple of facts about this topology:

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There is a natural topology defined on the symmetric group on a countable set such as $\mathbb{N}$, the topology of pointwise convergence: two permutations g and h are close together they agree on a long initial segment of $\mathbb{N}$. Without going into details, here are a couple of facts about this topology:

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Thus, automorphism groups are complete metric spaces in their own right.

The Kechris–Pestov–Todorcevic theorem

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An application

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It turns out that this is the only way to achieve it. For suppose that $\mathcal{A}$ is a Fraïssé class, and M its Fraïssé limit. It is not hard to show that the set of linear orders of M is a compact Hausdorff space. By the KPT theorem, there is a linear order fixed by the automorphism group. This induces a linear order on each structure in the age of M.

The pseudo-arc

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We have to be careful about the word "typical". In a probability space this can mean "a set of measure 1", but here we don't have a measure. Instead we use a notion from Baire category: in a complete metric space, a set is residual if it contains a countable intersection of open dense subsets. Residual sets behave like complements of null sets: they are non-empty, meet every open set, and any two (or countably many) of them intersect in a residual set.

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The metric we use on closed subsets of the square is Hausdorff metric: two sets are within distance ϵ if every point of one is within distance ϵ from some point of the other.

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Moreover, the statement in the first paragraph remains true if we replace the unit square by the unit hypercube in $\mathbb{R}^n$ for any $n \geq 2$, or in Hilbert space.
Its topological definition might suggest that it cannot be constructed by discrete methods, but this is not so …

Turning the arrows round

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There is an additional feature. The inverse limit of a family of finite sets and maps is a subset of the Cartesian product of the finite sets, and so has a topology, induced from the Tychonoff product topology of the discrete topologies on the finite sets. It is not discrete, but is totally disconnected.

Constructing the pseudo-arc

Consider the class $\mathcal{P}$ of reflexive paths, graphs which consist of a finite path with a loop at each vertex. Irwin and Solecki show that $\mathcal{P}$ is a projective Fraïssé class, so has a projective Fraïssé limit P.

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They show further that the graph structure on P consists of isolated vertices and edges (with loops) only; thus, an equivalence relation with all equivalence classes of size 1 or 2.

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Using this, Solecki and Tsankov were able to give a new proof of Bing's theorem that the pseudo-arc has a transitive homeomorphism group.