Graphs defined on groups

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Algebra Seminar, University of St Andrews
21 October 2020

Notice

This is a preview of a talk I will be giving at Presidency University, Kolkata, next month, 90 minute lecture aimed at young researchers.
I have made some cuts for this version.
You are very welcome to think about some of these questions.
If you need more information, just ask!

Graphs and groups

"Graphs and groups" is a very big topic; there are links in both directions between these two areas. Let me begin by briefly mentioning two things that I will not be talking about.

- Every group is the automorphism group of a graph, as Frucht showed in 1939. Indeed, many restrictions can be put on the graph, such as specifying the valency, connectivity, or chromatic number, and the result remains true.
- Several of the sporadic simple groups were first constructed as groups of automorphisms of certain graphs; the Higman–Sims group was a famous example. (The graph had been constructed 12 years earlier by Dale Mesner, but he did not know about 3-designs and Mathieu groups, which make the job much easier, and he did not investigate the automorphism group.)

My topic is more specific. I will mostly talk about graphs whose vertex set is a group G, and where the graph reflects structural properties of G. There are three main areas:

- Cayley graphs, invariant under translation by group elements (and hence vertex-transitive);
- graphs defined more directly, and invariant under automorphisms of the group, such as the commuting graph or generating graph;
- other graphs related to the group (but not having the group elements as vertices) giving structural information, such as the prime graph or intersection graph.

Caveats

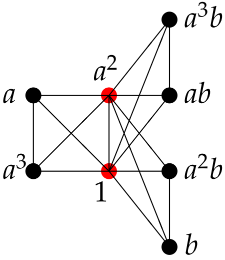
- Several of these graphs were defined first for semigroups. I will not consider semigroups here. Note that many of the problems I raise can be asked also for semigroups, and other interesting questions arise too.
- I also cannot cover infinite groups here, apart from an occasional brief remark, though there are many hard and interesting problems there.
- Reluctantly I also omit Cayley graphs: their study includes all of geometric group theory.

The commuting graph

I begin with the example which is easiest to define.
In a finite group, the relation of commuting between two elements ($gh = hg$) is symmetric, so we can use it to define a graph.
Let G be a group. The commuting graph of G is the graph with vertex set G, in which two vertices g and h are joined if $gh = hg$.
WARNING: I will slightly modify the definition shortly …
As defined, the graph has a loop at each vertex, since any element commutes with itself. Also, elements in the centre $Z(G)$ of G are joined to all vertices.

Examples

If G is abelian, then its commuting graph is complete.
The two non-abelian groups of order 8 (the dihedral and quaternion groups) have isomorphic commuting graphs, as shown. The groups are $\langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $\langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$. Elements of the centre are coloured red. Loops have not been drawn.



Random walk on the commuting graph

One remarkable property of the commuting graph is due, essentially, to Mark Jerrum:

Theorem

The limiting distribution of the random walk on the commuting graph of G is uniform on the conjugacy classes.

The random walk moves at discrete time steps; at each step, if you are at a vertex, you choose a neighbouring vertex uniformly at random and move there. I will not explain here the theory of random walks. If you know it, the proof is a simple exercise; if you don't, please take it on trust!

The conclusion of the theorem means that the probability of being at a vertex g at time t tends to a limit as $t \to \infty$, the limit being inversely proportional to the size of the conjugacy class containing g. So you are equally likely to be in any class.

This can be used to find elements in classes too small to be found by random search.

General questions

What questions should we ask about a graph? Here are some:

- Does it have isolated vertices, or vertices joined to all others?
- Is it connected? If so, what is its diameter? If not, how many connected components does it have, and what are their diameters?
- Any other graph-theoretic question, e.g. is it Hamiltonian, what is its chromatic number, what is its domination number, is it perfect, etc.
- Does the graph determine the group, or some of its properties?

In the case of the commuting graph, connectedness has a very simple positive answer: since elements of the centre are joined to everything, the diameter is at most 2.

So to make the question less trivial, we redefine the commuting graph so that the vertex set is $G \setminus Z(G)$.

Diameter of the commuting graph

To reiterate: we define the commuting graph to have vertex set $G \setminus Z(G)$; $x \sim y$ if $xy = yx$.

Theorem (Giudici and Parker)

There is no upper bound for the diameter of the commuting graph of a finite group; for any given d there is a 2-group whose commuting graph is connected with diameter greater than d.

On the other hand:

Theorem (Morgan and Parker)

Suppose that the finite group G has trivial centre. Then every connected component of its commuting graph has diameter at most 10.

Is a non-abelian group determined by its commuting graph?

The answer is no, in general. The two non-abelian groups of order 8 (the dihedral and quaternion groups) each have centre of order 2, and non-central elements have centraliser of order 4. So in each case the commuting graph consists of three disjoint edges. So the question is:

Question

Which non-abelian groups G are determined up to isomorphism by their commuting graphs?

One could ask something weaker. Does the commuting graph of G determine the order of G? (Note that the number of vertices of the graph is $|G| - |Z(G)|$.)
I do not know a counterexample to this. It is known to be true for many groups.

The prime graph

A graph with connections to the commuting graph is the prime graph, or Gruenberg–Kegel graph, of the group G. Its vertices are the prime divisors of $|G|$; there is an edge from p to q if and only if G contains an element of order pq.

Gruenberg and Kegel introduced this graph, in an unpublished manuscript in 1975, in the study of integral representations of groups. They noted that groups whose prime graph is disconnected have a very restricted structure. This was worked out in detail by Williams in 1981 except for groups of Lie type in characteristic 2 (the work was completed and some errors corrected by Kondrat'ev and Mazurov in 2000).

Proposition

Let G be a finite group with $Z(G) = 1$. Then the commuting graph of G is connected if and only if the prime graph is connected.

The proof is elementary and does not use the structure of groups with disconnected prime graph.

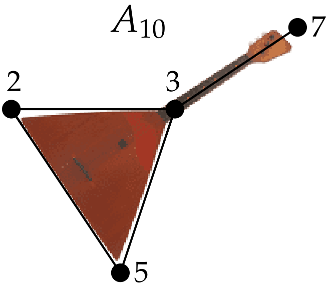
About the prime graph

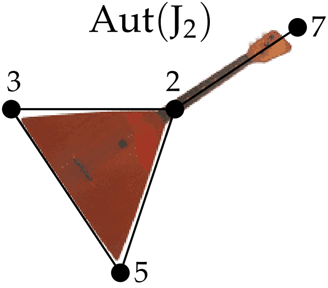
There has been a lot of research on the prime graph of a finite group. Much of this is due to group theorists in Yekaterinberg and Novosibirsk.

Some of the questions considered are:

- ☞ Which groups are characterised by their prime graphs?
- ☞ Which groups are characterised by their labelled prime graphs, where the vertices are labelled with the corresponding primes, and how many different labellings can a given graph have?

To mention just one example: the paw, or balalaika, consists of a triangle with a pendant vertex. Among groups whose prime graph is isomorphic to the paw are the alternating group A_{10} and the automorphism group of the sporadic Janko group J_2.





Note that the same primes occur but 2 and 3 swap places.

The enhanced power graph

I will reverse the historical order here; the power graph was defined before the enhanced power graph (and it was defined for semigroups before groups).

Also, I will change my conventions once again. In almost all the graphs I will define, the identity element of the group plays a special role. So from now on, I will always take the vertex set to be $G \setminus \{1\}$, the set of non-identity elements.

In the enhanced power graph, we define x and y to be adjacent if there is a vertex z such that both x and y are powers of z.

This can be re-phrased: x and y are joined if and only if the subgroup $\langle x, y \rangle$ generated by x and y is cyclic.

With this in mind, note that x and y are joined in the commuting graph if and only if $\langle x, y \rangle$ is abelian.

Thus the enhanced power graph is a spanning subgraph of the commuting graph.

The power graph

The directed power graph of a group G is defined as follows: the vertex set is $G \setminus \{1\}$; there is an arc $x \to y$ if y is a power of x. Note that there may be arcs in both directions: this holds if and only each of x and y is a power of the other, that is, $\langle x \rangle = \langle y \rangle$.

The power graph is obtained from the directed power graph by ignoring the directions on the edges (and double edges where they occur): that is, $x \sim y$ if and only if one of x and y is a power of the other.

We see that the power graph is a spanning subgraph of the enhanced power graph.

Some observations

Different groups may have isomorphic power graphs. For example, let G be a group with exponent 3 (of order 3^d, say). Then the power graph of G consists of $(3^d - 1)/2$ disjoint edges. So any other group of exponent 3 with the same order has isomorphic power graph.

Things are even worse in the infinite case. The Prüfer group C_{p^∞} has the property that its power graph is a countable complete graph; we cannot even detect the prime p.

From now on I stick to the finite case. Here are some observations.

- The power graph of G is complete if and only if G is cyclic of prime power order.
- Otherwise, the vertex g is joined to all others if and only if either G is cyclic and g is a generator, or G is a generalized quaternion group and g the central involution.

Determining the directions

The power graph of G may not determine G. But does it determine the directed power graph?

It does not determine it uniquely. For example, if G is cyclic of prime power order, then the power graph is complete, and there is no way to detect the generators.

But the following holds:

Theorem

The power graph of a finite group G determines the directed power graph of G up to isomorphism.

Hence the power graph determines the enhanced power graph up to isomorphism.

A variation

We saw that the enhanced power graph and the commuting graph have similar definitions: adjacency of x and y can be defined in terms of the subgroup $\langle x, y \rangle$ they generate: cyclic for the enhanced power graph, abelian for the commuting graph. This suggests a raft of open questions.

Question

Let P be a property of groups. For a finite group G, define a graph as follows: the vertex set is $G \setminus \{1\}$, and x and y are joined if and only if the subgroup $\langle x, y \rangle$ has property P.
For different choices of P, what can be said about this graph?

The properties of nilpotence or solubility might be good places to start. Essentially nothing is known; the field is open!

The generating graph

Another variant is as follows:
The generating graph of G is the graph with vertex set $G \setminus \{1\}$, in which x and y are joined if and only if $\langle x, y \rangle = G$.
Note that this graph has no edges if G cannot be generated by two elements; it has loops if and only if G is cyclic. These cases are not very interesting.
This graph is of great interest. Since the Classification of Finite Simple Groups (CFSG), it is known that every non-abelian finite simple group can be generated by two elements.
Moreover, a number of results assert that there is a great deal of freedom in choosing the two generators. For example, any non-identity element lies in a 2-element generating set.

The best result is in a recent preprint by Burness, Guralnick and Harper:

Theorem

For a finite group G, the following three conditions are equivalent:

- ☞ *the generating graph has no isolated vertices;*
- ☞ *any two vertices of the generating graph have a common neighbour;*
- ☞ *every proper quotient of G is cyclic.*

So in particular the first two properties hold for the generating graph of a non-abelian simple group.

A hierarchy

Part of the reason for choosing the vertex set of these graphs to be $G \setminus \{1\}$ is to allow them to be compared.

Given a group G, consider the following graphs:

- the null graph on $G \setminus \{1\}$;
- the power graph;
- the enhanced power graph;
- the commuting graph;
- the non-generating graph (the complement of the generating graph);
- the complete graph.

Each of these graphs is a spanning subgraph of the next, except possibly for the commuting and non-generating graphs.

Indeed the commuting graph is a spanning subgraph of the non-generating graph if G is non-abelian.

Equality

Question

When are two consecutive graphs in the hierarchy equal?

- The non-generating graph is complete if and only if G is not 2-generated.
- For non-abelian groups G, the commuting graph is equal to the non-generating graph if and only if G is minimal non-abelian.
- The enhanced power graph is equal to the commuting graph if and only if G contains no subgroup $C_p \times C_p$ for p prime.
- The power graph is equal to the enhanced power graph if and only if G contains no subgroup $C_p \times C_q$, for p and q distinct primes (i.e. the prime graph of G is a null graph).
- The power graph is null if and only if G is an elementary abelian 2-group.

Next steps

These results on the hierarchy suggest a couple of questions.

Question

Do similar results hold for infinite groups?

Aalipour *et al.* have some results for infinite soluble groups with power graph equal to commuting graph.

Question

If successive graphs in the hierarchy are not equal, what can be said about their difference?

The difference between the complete graph and the non-generating graph is, of course, the generating graph, about which a lot is known (as already mentioned).

The difference between the non-generating graph and the commuting graph (for non-abelian groups) has been considered by Saul Freedman. His thesis contains detailed results about connected components and diameter of these graphs, which I cannot summarise here. The results include a complete analysis for finite nilpotent groups.

For the next two cases, the difference between commuting graph and enhanced power graph, or enhanced power graph and power graph, nothing is known, the field is wide open.

Perfectness

One of the most important properties of the power graph is that it is perfect.

A graph is perfect if every induced subgraph has clique number equal to chromatic number. The Strong Perfect Graph Theorem (proving a conjecture of Berge) shows that a graph is perfect if and only if it does not contain as induced subgraph a cycle of odd length greater than 3 or the complement of one.

It is known that many decision problems such as clique size and chromatic number, which are hard for general graphs, are polynomial-time for perfect graphs (using semidefinite programming).

Perfect graphs also contain many other important graph classes including bipartite graphs and their complements, comparability graphs of partial orders and their complements, interval graphs, cographs, chordal graphs, etc.

Theorem

A power graph is perfect.

The proof works not only for groups but for semigroups and even for power-associative magmas; it also works for infinite objects, if we say that an infinite graph is perfect if all its finite subgraphs have clique number equal to chromatic number. Briefly: the directed power graph is a partial preorder, a reflexive and transitive relation, and the power graph is its comparability graph. We can turn a partial preorder into a partial order without changing the comparability graph: the relation $\equiv$ defined by $x \equiv y$ if $x \to y$ and $y \to x$ is an equivalence relation; now refine the order by taking any total order on each equivalence class. Now the easy direction in Dilworth's Theorem says that the comparability graph of a partial order is perfect.

A problem

The enhanced power graph and the commuting graph can fail to be perfect. Consider, for example, the commuting graph of S_5, and look at the transpositions $(1,2)$, $(3,4)$, $(5,1)$, $(2,3)$ and $(4,5)$. The induced subgraph of the commuting graph is a 5-cycle, with clique number 2 and chromatic number 3.

It is easy to modify this example to deal with the enhanced power graph, replacing transpositions with cycles of pairwise coprime lengths.

Problem

- *Which groups have perfect enhanced power graph?*
- *Which groups have perfect commuting graph?*

The groups considered earlier, where the enhanced power graph or commuting graph are equal to the power graph, have this property. What others exist?

Other properties

Pallabi Manna, Ranjit Mehatari and I are conducting a study of groups whose power graphs belong to various well-studied classes defined by forbidden induced subgraphs. We have complete results for nilpotent groups, and in some cases for arbitrary finite groups.

A threshold graph is a graph which can be built from a single vertex by the two operations of adding an isolated vertex and adding a vertex joined to everything.

A split graph is a graph whose vertex set is the disjoint union of a complete graph and a null graph (with arbitrary edges between).

Every threshold graph is split, and every split graph is perfect. Both classes have been widely investigated, have good algorithmic properties, and have applications in computer science and elsewhere.

Which power graphs are threshold or split?

Theorem

For a finite group G, the following conditions are equivalent:

- ☞ *the power graph of G is a threshold graph;*
- ☞ *the power graph of G is a split graph;*
- ☞ *G is cyclic of prime power order, or an elementary abelian or dihedral 2-group, or a cyclic group of order $2p$, or a dihedral group of order $2p^n$ or $4p$, where p is an odd prime and $n \geq 1$.*

In fact these groups are precisely those whose power graphs have no induced subgraph consisting of two disjoint edges.

A challenge

Problem

Which if any of these results can be extended (in some form) to semigroups?