Graphs defined on groups

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Notices

- ☞ Questions welcome!
- ☞ There is a selected bibliography at the end of the slides.
- ☞ A "handout" version of the slides is available: please ask!

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Graphs and groups

"Graphs and groups" is a very big topic; there are links in both directions between these two areas. Let me begin by briefly mentioning two things that I will not be talking about.

- ☞ Every group is the automorphism group of a graph, as Frucht showed in 1939. Indeed, many restrictions can be put on the graph, such as specifying the valency, connectivity, or chromatic number, and the result remains true.
- ☞ Several of the sporadic simple groups were first constructed as groups of automorphisms of certain graphs; the Higman–Sims group was a famous example. (The graph had been constructed 12 years earlier by Dale Mesner, but he did not know about 3-designs and Mathieu groups, which make the job much easier, and he did not investigate the automorphism group.)

My topic is more specific. I will mostly talk about graphs whose vertex set is a group G, and where the graph reflects structural properties of G. There are three main areas:

- Cayley graphs, invariant under translation by group elements (and hence vertex-transitive);
- graphs defined more directly, and invariant under automorphisms of the group, such as the commuting graph or generating graph;
- other graphs related to the group (but not having the group elements as vertices) giving structural information, such as the Gruenberg–Kegel graph or the intersection graph.

Caveats

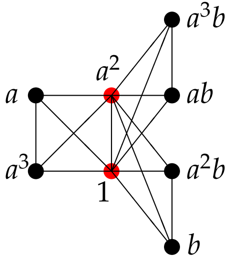
- Several of these graphs were defined first for semigroups. I will not consider semigroups here. Note that many of the problems I raise can be asked also for semigroups, and other interesting questions arise too.
- I also cannot cover infinite groups here, apart from an occasional brief remark, though there are many hard and interesting problems there.
- Reluctantly I also omit Cayley graphs: their study includes all of geometric group theory.

The commuting graph

I begin with the example which is easiest to define.
In a finite group, the relation of commuting between two elements ($gh = hg$) is symmetric, so we can use it to define a graph.
Let G be a group. The commuting graph of G is the graph with vertex set G, in which two vertices g and h are joined if $gh = hg$.
WARNING: I will slightly modify the definition shortly ...
As defined, the graph has a loop at each vertex, since any element commutes with itself. Also, elements in the centre $Z(G)$ of G are joined to all vertices.

Examples

If G is abelian, then its commuting graph is complete.
The two non-abelian groups of order 8 (the dihedral and quaternion groups) have isomorphic commuting graphs, as shown. The groups are $\langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $\langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$. Elements of the centre are coloured red. Loops have not been drawn.



Random walk on the commuting graph

One remarkable property of the commuting graph is due, essentially, to Mark Jerrum:

Theorem

The limiting distribution of the random walk on the commuting graph of G is uniform on the conjugacy classes.

The random walk moves at discrete time steps; at each step, if you are at a vertex, you choose a neighbouring vertex uniformly at random and move there. I will not explain here the theory of random walks. If you know it, the proof is a simple exercise; if you don't, please take it on trust!
The conclusion of the theorem means that the probability of being at a vertex g at time t tends to a limit as $t \to \infty$, the limit being inversely proportional to the size of the conjugacy class containing g. So you are equally likely to be in any conjugacy class.

Application

There are groups in which some conjugacy classes are much smaller than others. (For a simple example, consider the symmetric group S_n. The transpositions form a conjugacy class of size $n(n-1)/2$, while the n-cycles form a class of size $(n-1)!$. Very often, interesting elements lie in small classes. In computational group theory, it may happen that we have a group, and wish to find a representative of a very small conjugacy class. Simply choosing elements at random will be very inefficient, like looking for a needle in a haystack. However, a random walk on the commuting graph "amplifies" the small classes and makes it easier to find elements in these classes.

General questions

What questions should we ask about a graph? Here are some:

☞ Does it have isolated vertices, or vertices joined to all others?

☞ Is it connected? If so, what is its diameter? If not, how many connected components does it have, and what are their diameters?

☞ Any other graph-theoretic question, e.g. is it Hamiltonian, what is its chromatic number, what is its domination number, is it perfect, etc.

☞ Does the graph determine the group, or some of its properties?

In the case of the commuting graph, connectedness has a very simple positive answer: since elements of the centre are joined to everything, the diameter is at most 2.

So to make the question less trivial, we redefine the commuting graph so that the vertex set is $G \setminus Z(G)$.

Diameter of the commuting graph

To reiterate: we define the commuting graph to have vertex set $G \setminus Z(G)$; $x \sim y$ if $xy = yx$.

Theorem (Giudici and Parker)

There is no upper bound for the diameter of the commuting graph of a finite group; for any given d there is a 2-group whose commuting graph is connected with diameter greater than d.

On the other hand:

Theorem (Morgan and Parker)

Suppose that the finite group G has trivial centre. Then every connected component of its commuting graph has diameter at most 10.

Is a non-abelian group determined by its commuting graph?

The answer is no, in general. The two non-abelian groups of order 8 (the dihedral and quaternion groups) each have centre of order 2, and non-central elements have centraliser of order 4. So in each case the commuting graph consists of three disjoint edges. So the question is:

Question

Which non-abelian groups G are determined up to isomorphism by their commuting graphs?

One could ask something weaker. Does the commuting graph of G determine the order of G? (Note that the number of vertices of the graph is $|G| - |Z(G)|$.)
I do not know a counterexample to this. It is known to be true for many groups.

The Gruenberg–Kegel graph

A graph with connections to the commuting graph is the Gruenberg–Kegel graph of the group G, also known as the prime graph. Its vertices are the prime divisors of $|G|$; there is an edge from p to q if and only if G contains an element of order pq.

Gruenberg and Kegel introduced this graph, in an unpublished manuscript in 1975, in the study of integral representations of groups. They noted that groups whose GK graph is disconnected have a very restricted structure. This was worked out in detail by Williams in 1981 except for groups of Lie type in characteristic 2 (the work was completed by Kondrat'ev in 1989, and some errors corrected by Kondrat'ev and Mazurov in 2000).

The GK graph and the commuting graph

Proposition

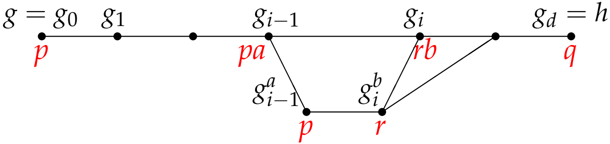
Let G be a finite group with $Z(G) = 1$. Then the commuting graph of G is connected if and only if the GK graph is connected.

The proof does not use the Classification of Finite Simple Groups, or even the structure of groups with disconnected GK graph. I outline it on the next three slides.

Proof

Suppose first that $Z(G) = 1$ and the commuting graph is connected. Let p and q be primes dividing $|G|$. Choose elements g and h of orders p and q respectively, and suppose their distance in the commuting graph is d. We show by induction on d that there is a path from p to q in the GK graph. If $d = 1$, then g and h commute, so gh has order pq, and p is joined to q.

So assume the result for distances less than d, and let $g = g_0, \ldots, g_d = h$ be a path from g to h.
Let i be mimimal such that p does not divide the order of g_i (so $i > 0$). Now some power of g_{i-1}, say g_{i-1}^a, has order p, while a power g_i^b of g_i has prime order $r \neq p$.



Orders written in red under the vertices

The distance from g_i^b to g_d is at most $d - i < d$, so there is a path from r to q in the GK graph. But g_{i-1}^a and g_i^b commute, so p is joined to r.

For the converse, assume that the GK graph is connected.
Note first that for every non-identity element g, some power of g has prime order, so it suffices to show that all elements of prime order lie in the same connected component of the commuting graph. Also, since a non-trivial p-group has non-trivial centre, the non-identity elements of any Sylow subgroup lie in a single connected component.
Let C be a connected component. Connectedness of the GK graph shows that C contains a Sylow p-subgroup for every prime p dividing $|G|$. Also, every element of C, acting by conjugation, fixes C. It follows that the normaliser of C is G, and hence that C contains every Sylow subgroup of G, and thus contains all elements of prime order, as required.

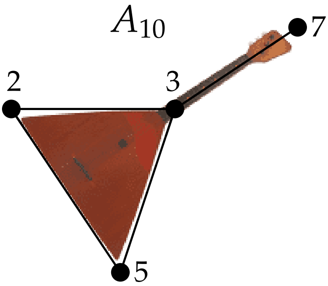
About the GK graph

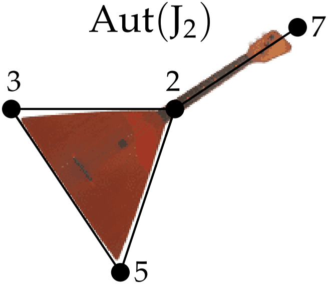
There has been a lot of research on the GK graph of a finite group. Much of this is due to group theorists in Yekaterinburg and Novosibirsk.

Some of the questions considered are:

- Which groups are characterised by their GK graphs?
- Which groups are characterised by their labelled GK graphs, where the vertices are labelled with the corresponding primes, and how many different labellings can a given graph have?

To mention just one example: the paw, or balalaika, consists of a triangle with a pendant vertex. Among groups whose GK graph is isomorphic to the paw are the alternating group A_{10} and the automorphism group of the sporadic Janko group J_2.





Note that the same primes occur but 2 and 3 swap places.

The enhanced power graph

I will reverse the historical order here; the power graph was defined before the enhanced power graph.
Also, I will change my conventions once again. In almost all the graphs I will define, the identity element of the group plays a special role. So from now on, I will always take the vertex set to be $G \setminus \{1\}$, the set of non-identity elements.
In the enhanced power graph, we define x and y to be adjacent if there is a vertex z such that both x and y are powers of z.
This can be re-phrased: x and y are joined if and only if the subgroup $\langle x, y\rangle$ generated by x and y is cyclic.
With this in mind, note that x and y are joined in the commuting graph if and only if $\langle x, y\rangle$ is abelian.
Thus the enhanced power graph is a spanning subgraph of the commuting graph.

The power graph

The directed power graph of a group G is defined as follows: the vertex set is $G \setminus \{1\}$; there is an arc $x \to y$ if y is a power of x. Note that there may be arcs in both directions: this holds if and only each of x and y is a power of the other, that is, $\langle x \rangle = \langle y \rangle$.

The power graph is obtained from the directed power graph by ignoring the directions on the edges (and double edges where they occur): that is, $x \sim y$ if and only if one of x and y is a power of the other.

We see that the power graph is a spanning subgraph of the enhanced power graph.

Some observations

Different groups may have isomorphic power graphs. For example, let G be a group with exponent 3 (of order 3^d, say). Then the power graph of G consists of $(3^d - 1)/2$ disjoint edges. So any other group of exponent 3 with the same order has isomorphic power graph.

Things are even worse in the infinite case. The Prüfer group C_{p^∞} has the property that its power graph is a countable complete graph; we cannot even detect the prime p.

From now on I stick to the finite case. Here are some observations.

- The power graph of G is complete if and only if G is cyclic of prime power order.
- Otherwise, the vertex g is joined to all others if and only if either G is cyclic and g is a generator, or G is a generalized quaternion group and g the central involution.

Determining the directions

The power graph of G may not determine G. But does it determine the directed power graph?
It does not determine it uniquely. For example, if $G = C_6$, the power graph has five vertices, two with valency 4 (these are the generators), two with valency 3 (the elements of order 3), and one of valency 2 (the element of order 2). But we cannot decide which element of order 3 is the square of which element of order 6.
But the following holds:

Theorem

The power graph of a finite group G determines the directed power graph of G up to isomorphism.

Hence the power graph determines the enhanced power graph up to isomorphism.

A variation

We saw that the enhanced power graph and the commuting graph have similar definitions: adjacency of x and y can be defined in terms of the subgroup $\langle x, y \rangle$ they generate: cyclic for the enhanced power graph, abelian for the commuting graph. This suggests a raft of open questions.

Question

Let P be a property of groups. For a finite group G, define a graph as follows: the vertex set is $G \setminus \{1\}$, and x and y are joined if and only if the subgroup $\langle x, y \rangle$ has property P.
For different choices of P, what can be said about this graph?

The properties of nilpotence or solubility might be good places to start. Essentially nothing is known; the field is open!

The generating graph

Another variant is as follows:
The generating graph of G is the graph with vertex set $G \setminus \{1\}$, in which x and y are joined if and only if $\langle x, y \rangle = G$.
Note that this graph has no edges if G cannot be generated by two elements; it has loops if and only if G is cyclic. These cases are not very interesting.
This graph is of great interest. Since the Classification of Finite Simple Groups (CFSG), it is known that every non-abelian finite simple group can be generated by two elements.
Moreover, a number of results assert that there is a great deal of freedom in choosing the two generators. For example, any non-identity element lies in a 2-element generating set.

The best result is in a recent preprint by Burness, Guralnick and Harper:

Theorem

For a finite group G, the following three conditions are equivalent:

- *the generating graph has no isolated vertices;*
- *any two vertices of the generating graph have a common neighbour;*
- *every proper quotient of G is cyclic.*

So in particular the first two properties hold for the generating graph of a non-abelian simple group.

A hierarchy

Part of the reason for choosing the vertex set of these graphs to be $G \setminus \{1\}$ is to allow them to be compared.

Given a group G, consider the following graphs:

- ☞ the null graph on $G \setminus \{1\}$;
- ☞ the power graph;
- ☞ the enhanced power graph;
- ☞ the commuting graph;
- ☞ the non-generating graph (the complement of the generating graph);
- ☞ the complete graph.

Each of these graphs is a spanning subgraph of the next, except possibly for the commuting and non-generating graphs.

Indeed the commuting graph is a spanning subgraph of the non-generating graph if G is non-abelian.

Equality

Question

When are two consecutive graphs in the hierarchy equal?

This question has been answered completely for finite groups. In reverse order,

- The non-generating graph is complete if and only if G is not 2-generated.
- For non-abelian groups G, the commuting graph is equal to the non-generating graph if and only if G is minimal non-abelian, that is, G is non-abelian but every proper subgroup is abelian.

The minimal non-abelian groups were determined by Miller and Moreno in 1903. There are two types; one has prime power order, and the other has order $p^m q^n$, where p and q are primes and q is a primitive divisor of $p^m - 1$.

The next two cases are due to Aalipour, Akbari, Cameron, Nikandish and Shaveisi.

> The commuting graph is equal to the enhanced power graph if and only if G contains no subgroup isomorphic to $C_p \times C_p$ for p prime; equivalently, the Sylow subgroups of G are cyclic or generalized quaternion.

The groups G which occur here are either cyclic p-groups for some prime p, or have the property that $O(G)$ (the largest normal subgroup of G of odd order) is metacyclic, and $G/O(G)$ is a group with a unique involution (so the quotient by this involution is $\mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$ for p an odd prime power, or a cyclic or dihedral 2-group).

☞ The enhanced power graph is equal to the power graph if and only if G contains no cyclic subgroup of order pq, where p and q are distinct primes.

This condition says that the GK graph of G is a null graph. Using results on groups whose prime graph is disconnected, it is possible to study these. There are some interesting examples, including the simple groups A_5, A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,4)$ and $\mathrm{Sz}(8)$. Natalia Maslova I hope to publish the complete classification soon.
The last two classes are rather complicated, but it is simpler to give the list of finite groups with power graph equal to commuting graph. These are cyclic groups of prime power order, generalized quaternion groups, and semidirect products of C_{p^a} by C_{q^b} where p and q are primes, $q^b \mid p-1$ and C_{q^b} acts faithfully on C_{p^a}.

Finally, and trivially,

☞ the power graph of G is null if and only if G is the trivial group; and the reduced power graph is null if and only if G is an elementary abelian 2-group.

For it is clear that if there is an element of order greater than 2 in the group, then there is an edge in the power graph.

Next steps

These results on the hierarchy suggest a couple of questions.

Question

Do similar results hold for infinite groups?

Aalipour *et al.* have some results for infinite soluble groups with power graph equal to commuting graph.

Question

If successive graphs in the hierarchy are not equal, what can be said about their difference?

The difference between the complete graph and the non-generating graph is, of course, the generating graph, about which a lot is known (as already mentioned).

The difference between the non-generating graph and the commuting graph (for non-abelian groups) has been considered by Saul Freedman. His thesis contains detailed results about connected components and diameter of these graphs, which I cannot summarise here. The results include a complete analysis for finite nilpotent groups.
For the next two cases, the difference between commuting graph and enhanced power graph, or enhanced power graph and power graph, nothing is known, the field is wide open.
A further question:

Question

Does the random walk on any of these graphs have interesting properties?

Final remark on the hierarchy

For all the graphs $\Gamma(G)$ in the hierarchy given earlier except the non-generating graph, the following property holds:

> The induced subgraph of $\Gamma(G)$ on any subgroup H of G is equal to $\Gamma(H)$.

This is false for the non-generating graph. In that case, the induced subgraph on H is complete, since two elements of H generate a proper subgroup of G. Moreover, these complete subgraphs on proper subgroups of G cover all the edges of the graph.
Similarly, in the commuting graph (resp., the enhanced power graph), the induced subgraphs on the abelian (resp. cyclic) subgroups are complete and cover all edges.

The intersection graph

A related graph that has been investigated is the intersection graph of a group (usually but not always assumed finite). The vertices are not elements, but non-trivial proper subgroups of G; two vertices are adjacent if their intersection is non-trivial. This is related to the generating graph because two elements are adjacent in the non-generating graph if and only if they are contained in a proper subgroup, as noted on the preceding slide.

Three remarks:

☞ The intersection graph has at least one vertex if and only if G is not cyclic of prime order.

☞ In a finite group, every proper subgroup is contained in a maximal subgroup; so, if any two maximal subgroups lie at distance at most d, then the diameter of the graph is at most $d + 2$.

☞ Suppose that G is not a dihedral group. Then any two subgroups of even order lie at distance at most 2; for each contains an involution, and two involutions generate a dihedral group.

Diameter

The intersection graph was introduced by Csákány and Pollák in 1969. For non-simple groups, they determined when the graph is connected, and showed that in these cases its diameter is at most 4.

For simple groups, Shen proved connectedness in 2010; in the same year Herzog, Longobardi and Maj bounded the diameter by 64; and Ma reduced the bound to 28 in 2016. This year, Freedman showed:

Theorem

For a non-abelian finite simple group G, the intersection graph has diameter at most 5. The bound 5 is attained by the Baby Monster group; any other group achieving the bound must be a unitary group $\mathrm{PSU}(n, q)$ *with n prime.*

Question

It is known that for unitary groups the diameter can be 3, 4 or 5. Determine which groups realise each possible value.

Perfectness

One of the most important properties of the power graph is that it is perfect. (A graph is perfect if every induced subgraph has clique number equal to chromatic number.) The proof works not only for groups but for semigroups, and indeed for all power-associative magmas; and it holds for infinite groups etc. if we understand an infinite graph to be perfect if all its finite induced subgraphs are.

The reason is that the directed power graph, with a loop at each vertex, is a partial preorder, that is, a reflexive and transitive relation; and the power graph is its comparability graph, that is, two vertices are adjacent if and only if they are related in the preorder.

Theorem

The comparability graph of a partial preorder is perfect.

Proof

Let $\to$ be a partial preorder on X. Define a relation $\equiv$ by the rule that $x \equiv y$ if and only if $x \to y$ and $y \to x$. Then $\equiv$ is an equivalence relation. Now define a relation $\leq$ on the equivalence classes by the rule that $[x] \leq [y]$ if and only if $x' \to y'$ for some (and hence every) $x' \in [x]$ and $y' \in [y]$. Then $\leq$ is a partial order.
Now we refine the partial preorder to a partial order by taking an arbitrary total ordering of each $\equiv$-class. This does not change the comparability graph.
By the easy half of Dilworth's Theorem, the comparability graph of a partial order is perfect.

Corollary

The power graph of a group (or semigroup, or even power-associative magma) is perfect.

A problem

The enhanced power graph and the commuting graph can fail to be perfect. Consider, for example, the commuting graph of S_5, and look at the transpositions $(1,2)$, $(3,4)$, $(5,1)$, $(2,3)$ and $(4,5)$. The induced subgraph of the commuting graph is a 5-cycle, with clique number 2 and chromatic number 3.
It is easy to modify this example to deal with the enhanced power graph, replacing transpositions with cycles of pairwise coprime lengths.

Problem

- *Which groups have perfect enhanced power graph?*
- *Which groups have perfect commuting graph?*

The groups considered earlier, where the enhanced power graph or commuting graph are equal to the power graph, have this property. What others exist? Britnell and Gill have some results on perfectness of the commuting graph.

Products of graphs

The strong product of two graphs X and Y is the graph whose vertex set is the Cartesian product $V(X) \times V(Y)$, with an edge from (x_1, y_1) to (x_2, y_2) if and only if x_1 is equal or adjacent to x_2, and y_1 is equal or adjacent to y_2, but we don't have equality in both. It is denoted by $X \boxtimes Y$ (the symbol represents the corresponding product of two copies of K_2).
Now it is easy to show that

- ☞ the commuting graph of $G \times H$ is the strong product of the commuting graphs of G and H;
- ☞ if the orders of G and H are coprime, then the enhanced power graph of $G \times H$ is the strong product of the enhanced power graphs of G and H.

Perfectness of the enhanced power graph

The enhanced power graphs of finite nilpotent groups have clique number = chromatic number. When are they perfect?

- ☞ A nilpotent group is the direct product of its Sylow subgroups, whose orders are powers of distinct primes.
- ☞ For a group of prime power order, the enhanced power graph is equal to the power graph, which is perfect.
- ☞ The complement of a perfect graph is perfect (the weak perfect graph theorem of Lovász).
- ☞ The complement of the strong product of two graphs is the categorical product of their complements.
- ☞ Ravindra and Parthasarathy determined conditions for the categorical product of two graphs to be perfect.

Once we have done nilpotent groups, we may get information about the GK-graphs of arbitrary groups whose enhanced power graphs are perfect.

Other properties

A large number of other graph-theoretic properties are listed in the survey article by Abawajy, Kelarev and Chowdhury. Some extensions to infinite groups are given by Aalipour *et al.*

For example, the clique number of the enhanced power graph of a finite group G is equal to the largest order of an element of G. (This may not be the same as the exponent of G.) This holds also for infinite torsion groups of bounded exponent.

Pallabi Manna, Ranjit Mehatari and I are conducting a study of groups whose power graphs belong to various well-studied classes defined by forbidden induced subgraphs. We have complete results for nilpotent groups, which I describe on the next two slides.

Forbidden induced subgraphs

A cograph is a graph containing no 4-vertex path as induced subgraph. Cographs form the smallest nonempty class of graphs closed under complement and disjoint union.

Theorem

If G is a finite nilpotent group, then the power graph of G is a cograph if and only if G is either of prime power order, or a cyclic group whose order is the product of two distinct primes.

A chordal graph is a graph containing no chordless cycle of length greater than 3 as an induced subgraph.

Theorem

If G is a finite nilpotent group, then the power graph of G is chordal if and only if either G is of prime power order, or $|G|$ has two prime divisors and one of its Sylow subgroups is cyclic, the other of prime exponent.

Threshold and split graphs

A threshold graph is a graph containing no induced path or cycle on 4 vertices and no two disjoint unconnected edges. Threshold graphs form the smallest non-empty graph class closed under the operations of adding an isolated vertex or a vertex joined to all others.

A split graph is a graph whose vertex set is the disjoint union of a set inducing a complete graph and a set inducing a null graph. The split graphs form a class with nice algorithmic properties: for example a split graph can be recognised by its degree sequence. A graph is split if and only if it has no induced cycle of length 4 or 5 and no two disjoint unconnected edges.

We can determine all groups whose power graph is threshold or split.

Theorem

The following conditions on a finite group G are equivalent:

- *$P(G)$ is a threshold graph;*
- *$P(G)$ is a split graph;*
- *$P(G)$ contains no two disjoint unconnected edges;*
- *G is one of: a cyclic group of prime power order; an elementary abelian 2-group; a dihedral 2-group; a cyclic group of order $2p$, or a dihedral group of order $2p^d$ or $4p$, where p is an odd prime and $d \geq 1$.*

These are precisely the groups G having no pair (H, K) of subgroups such that both $H \setminus K$ and $K \setminus H$ contain elements of order greater than 2.

Some number theory

Here is an example. When is $P(\mathrm{PSL}(2,q))$ a cograph?
Let q be a power of 2. The power graph of $\mathrm{PSL}(2,q)$ is the disjoint union of copies of the power graph of an elementary abelian group of order q and cyclic groups of orders $q-1$ and $q+1$. (I assume here that the identity is removed.)
The case where one of these is a prime corresponds to Fermat and Mersenne primes, and the case where one is a prime power gives the Catalan equation, whose only solution is $3^2 = 1 + 2^3$.
To determine other groups $\mathrm{PSL}(2,q)$ for even q whose power graph is a cograph, we have to solve the following problem:

Problem

For which positive integers d is it the case that both $2^d - 1$ *and* $2^d + 1$ *are the product of at most two primes? In particular, are there infinitely many?*

The values of d up to 200 for which this holds are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.

Automorphism groups

The study of the automorphism groups of the graphs defined here is bedevilled by the fact that they have many "twins", pairs of vertices with the same neighbourhood.

For example, consider the generating graph of the alternating group A_5. Two elements which generate the same cyclic subgroup have the same neighbourhood. Thus we can find six disjoint sets of four vertices, and ten disjoint pairs of vertices, which are indistinguishable; so the automorphism group has a normal subgroup $(S_4)^6 \times (S_2)^{10}$ which is of no interest but simply slows down the computation.

So for questions about automorphisms, it may be sensible to factor out this normal subgroup before proceeding.

Automorphism groups of power graphs

A paper by Ashrafi, Gholami and Mehranian in 2017 considers this topic.

Let $\mathcal{F}$ be the class of groups obtained from the trivial group by the two operations "direct product" and "wreath product with a symmetric group".

Ashrafi *et al.* asked whether the automorphism group of the power graph of any finite group belongs to $\mathcal{F}$.

The answer is NO; the Mathieu group M_{11} provides a counterexample.

The power graph of M_{11}

The reduced power graph of M_{11} has the following connected components:

- 144 complete graphs of size 10, corresponding to elements of order 11;
- 396 complete graphs of size 4, corresponding to elements of order 5;
- a single connected component Δ on the remaining 4895 vertices.

If we take Δ, and first factor out the relation "same closed neighbourhood", and then factor out from the result the relation "same open neighbourhood", we obtain a connected graph on 1210 vertices whose automorphism group is M_{11}.

On the positive side, we have the following result:

Theorem

The automorphism group of any cograph belongs to $\mathcal{F}$.

This property does not characterise cographs: automorphism groups of trees belong to $\mathcal{F}$ for structural reasons, and automorphism groups of almost all random graphs belong to $\mathcal{F}$ for the stupid reason that almost all random graphs have trivial automorphism group.

Corollary

If G is a finite group whose power graph is a cograph, then the automorphism group of the power graph belongs to $\mathcal{F}$.

Problem

For which finite simple groups G is it true that $\mathrm{Aut}(G)$ is a homomorphic image of the automorphism group of the power graph of G?

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for your attention. Stay well!