Graphs defined on groups: Old and new connections

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Outline

There are three reasons for studying graphs whose vertex set is a group G and whose adjacency rule depends on the group structure. First, we may obtain some interesting and useful graphs by this means. Second, the graph may help us study the group. And finally, we meet some interesting things along the way …

In this talk I will discuss three such graphs: the commuting graph, the power graph, and the enhanced power graph.

Interestingly, it turns out that another much smaller graph, the Gruenberg–Kegel graph of the group, is intimately concerned with properties of all three of these graphs.

Finally I will define here for the first time a new graph, which I call the deep commuting graph, which involves Schur covers of the group G, and may help understand these.

The commuting graph

I begin with the example which is easiest to define. This goes back to the celebrated paper of Brauer and Fowler.

In a finite group, the relation of commuting between two elements ($gh = hg$) is symmetric, so we can use it to define a graph.

Let G be a group. The commuting graph of G is the graph with vertex set G, in which two vertices g and h are joined if $gh = hg$.

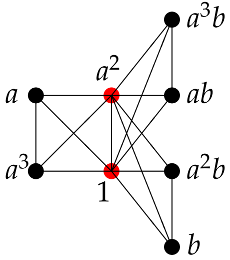
As defined, the graph has a loop at each vertex, since any element commutes with itself. Also, elements in the centre $Z(G)$ of G are joined to all vertices.

An application of the Orbit-counting Lemma shows that the proportion of all pairs of elements of G which commute (the number of directed edges in the commuting graph) to the total number of pairs is equal to the ratio of the number of conjugacy classes to the order of G (this ratio is sometimes called the commutativity degree or commuting probability of G).

Sketch: Let the group G act on itself by conjugation. Then the set of fixed points of x is the centraliser of x, the set of neighbours of x in the commuting graph. The orbits of G are the conjugacy classes.

Examples

If G is abelian, then its commuting graph is complete.
The two non-abelian groups of order 8 (the dihedral and quaternion groups) have isomorphic commuting graphs, as shown. The groups are $\langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $\langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$. Elements of the centre are coloured red. Loops have not been drawn.



Random walk on the commuting graph

One remarkable property of the commuting graph is due, essentially, to Mark Jerrum:

Theorem

The limiting distribution of the random walk on the commuting graph of G is uniform on the conjugacy classes.

In the preceding example, it is clear that the random walk spends longer at the red vertices (elements of the centre) than in the black vertices.
This property is used in computational group theory, for finding representatives of very small conjugacy classes in large groups: it "amplifies" the small classes to make them easier to find.

Connectedness and diameter

The commuting graph of G clearly is connected with diameter at most 2, since vertices in the centre of G are joined to all vertices.
To make the question more interesting, we define the reduced commuting graph of G to be the induced subgraph of the commuting graph on the set $G \setminus Z(G)$.
In our previous example of D_8 and Q_8, we see that in each case the reduced commuting graph consists of three connected components, each a single edge.
On the next slide I give two of the most significant results on this. It was conjectured that each connected component has bounded diameter; this turns out not to be the case.

Theorem (Giudici and Parker)

There is no upper bound for the diameter of the reduced commuting graph of a finite group; for any given d there is a 2-group whose commuting graph is connected with diameter greater than d.

On the other hand:

Theorem (Morgan and Parker)

Suppose that the finite group G has trivial centre. Then every connected component of its reduced commuting graph has diameter at most 10.

Can we decide whether or not the reduced commuting graph is connected? Sometimes yes; it is time to meet the next character in the story.

Enter the Gruenberg–Kegel graph

Let G be a finite group. The Gruenberg–Kegel graph of G is the graph whose vertex set is the set of prime divisors of $|G|$; there is an edge joining p to q if and only if G contains an element of order pq.

The graph was introduced by Gruenberg and Kegel in an unpublished manuscript in 1975 investigating integral representations of G, and in particular the decomposition of the augmentation ideal. In 1981, Williams (a student of Gruenberg) published their main result and began investigating simple groups; this was completed by Kondrat'ev in 1989, and some errors corrected by Kondrat'ev and Mazurov in 2000.

The Gruenberg–Kegel Theorem

Theorem

Let G be a finite group whose Gruenberg–Kegel graph is disconnected. Then either G is a Frobenius or 2-Frobenius group, or (with π the set of primes in the connected component containing 2) G is an extension of a nilpotent π-group by a simple group by a π-group.

A 2-Frobenius group is a group G with normal subgroups H and K with $H \leq K$ such that

- K is a Frobenius group with Frobenius kernel H;
- G/H is a Frobenius group with Frobenius kernel K/H.

The GK graph and the commuting graph

Theorem

Let G be a finite group with $Z(G) = 1$. Then the Gruenberg–Kegel graph of G is connected if and only if the reduced commuting graph is connected.

The proof is quite elementary; it does not use the Classification of Finite Simple Groups, or even the Gruenberg–Kegel theorem about groups whose GK graph is disconnected.

Things are more complicated for groups with non-trivial centre. Something can be said; but I will not discuss this further here.

The enhanced power graph

Now I move on to some further graphs defined on a group. I will reverse the historical order here and discuss the enhanced power graph (Aalipour *et al.* 2017) before the power graph (Kelarev and Quinn 1999).

The enhanced power graph of the finite group G has vertex set G; vertices x and y are joined if and only if there exists an element z such that both x and y are powers of z.

Note that x and y are joined in the enhanced power graph if and only if $\langle x, y \rangle$ is cyclic. Compare the commuting graph, where x and y are joined if and only if $\langle x, y \rangle$ is abelian. Thus the enhanced power graph is a spanning subgraph of the commuting graph (that is, it uses all the vertices and some of the edges).

Later in this talk I will introduce another graph lying between these two.

The power graph

This graph begins life as a directed graph. Let G be a finite group. The directed power graph of G is the graph with vertex set G, in which there is an arc from x to y if and only if y is a power of x.

The power graph is obtained simply by ignoring the directions: that is, x and y are joined if and only if one of them is a power of the other.

Thus the power graph is a spanning subgraph of the enhanced power graph.

Determining the directions

Can we recover the directions on the edges if we are given the power graph of a group?
Not in general. Given the cyclic group of order 6, there are three vertices joined to all others in the power graph (the identity and the two generators), and we cannot tell which of the three is the identity from the power graph. However, the following is true:

Theorem

The power graph of a finite group G determines the directed power graph up to isomorphism.

Corollary

The power graph of a finite group G determines the enhanced power graph up to isomorphism.

For $x \sim y$ in the enhanced power graph if and only if there exists z with $z \to x$ and $z \to y$ in the directed power graph.

Comparisons

If Γ_1, Γ_2 and Γ_3 denote the power graph, enhanced power graph, and commuting graph of a group G, we have seen that $E(\Gamma_1) \subseteq E(\Gamma_2) \subseteq E(\Gamma_3)$. When can two of these be equal?

Theorem

The enhanced power graph of G is equal to the commuting graph if and only if the Sylow subgroups of G are cyclic or generalized quaternion.

For the condition for equality (any two commuting elements generate a cyclic group) says precisely that G contains no subgroup $C_p \times C_p$ for prime p; now a theorem of Burnside gives the result.
Using known results on such groups, including the Gorenstein–Walter theorem, it is powwible to give a complete classification of such groups.

For the other comparison, the GK graph reappears:

Theorem

The power graph of G is equal to the enhanced power graph of G if and only if the Gruenberg–Kegel graph of G has no edges.

For the condition for equality is that if two elements generate a cyclic group then one is a power of the other. A cyclic group of order pq, for p and q distinct primes, would violate this.

From results on groups with disconnected GK-graph it is possible to extract a classification of all such groups. Natalia Maslova and I hope to publish the details shortly.

Cographs

Cographs form an important class of finite graphs which have a simple recursive construction and have good algorithmic properties.
The graph Γ is a cograph if it can be constructed from the one-vertex graph by the operations of disjoint union and complementation. Cographs form the smallest non-empty class of graphs closed under these two operations; they have the property that a cograph is connected if and only if its complement is disconnected. The smallest graph which is not a cograph is the 4-vertex path; thus a graph is a cograph if and only if it does not have the 4-vertex path as induced subgraph.
Pallabi Manna, Ranjit Mehatari and I showed:

Theorem

A finite nilpotent group G has the property that its power graph is a cograph if and only if either G has prime power order, or G is a cyclic group whose order is the product of two distinct primes.

We do not have a classification of finite groups whose power graph is a cograph. But we have a sufficient condition, and a necessary condition, as follows.

Theorem

- *Suppose that G is a finite group whose Gruenberg–Kegel graph has no edges. Then the power graph of G is a cograph.*
- *Suppose that G is a non-solvable finite group whose power graph is a cograph. Then every connected component of the Gruenberg–Kegel graph of G, except possibly the component containing the prime 2, has one or two vertices; if it has two vertices p and q, then p and q divide $|G|$ to the first power only.*

For the second part, we use a theorem of Williams. If σ is the vertex set of a connected component not containing 2 of the GK graph of a non-solvable group G, then G has a nilpotent Hall σ-subgroup; now the result follows from the result with Manna and Mehatari on nilpotent groups.

The complete classification of groups with power graph a cograph involves some intractible number-theoretic problems. For example, $\mathrm{PSL}(2, 2^m)$ has power graph a cograph if and only if each of $2^m + 1$ and $2^m - 1$ is either a prime power or the product of two distinct primes.

Which powers of 2 have this property? The question has a similar spirit to the existence of Fermat and Mersenne primes, but is not equivalent to either (though probably still rather difficult).

The list of values of m up to 200 for which the power graph of $\mathrm{PSL}(2, 2^m)$ is a cograph is 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.

Is the set of all such numbers finite or infinite?

Perfect graphs

Perfect graphs form another important graph class with good algorithmic properties, including many special classes including bipartite graphs, comparability graphs of partial orders, interval graphs, cographs, etc., and closed under complementation.

A graph is perfect if and only if every induced subgraph has clique number equal to chromatic number.

According to the strong perfect graph theorem, a graph is perfect if and only if it does not have an induced subgraph which is a cycle of odd length greater than 3 or the complement of one.

Perfectness of the power graph

The directed power graph is a partial preorder (a reflexive and transitive relation), and the power graph is its comparability graph.

According to (the easy direction of) Dilworth's Theorem, the comparability graph of a partial order is perfect. A small extra argument shows that the comparability graph of a partial preorder is perfect.

It is not necessary for this argument that G is a group; the argument works equally well for a semigroup, or indeed for any power-associative magma.

Question

For which groups is (a) the enhanced power graph, or (b) the commuting graph, a perfect graph?

Britnell and Gill have good results on commuting graphs.

Other questions

One can easily generate a raft of questions about these graphs, most of which are unanswered (and probably unstudied).

- What about other properties of these graphs, such as clique number, chromatic number, independence number, domination number, Hamiltonicity, etc.
- As well as the three graphs so far mentioned, what about their differences (the graph whose edges are edges of the commuting graph but not the enhanced power graph, or of the enhanced power graph but not the power graph)?
- What about infinite groups?
- What about semigroups?

Other graphs

Various other graphs on groups have been defined.

The generating graph of a group G has x and y joined by an edge if $\langle x, y \rangle = G$. It is easy to see that, if G is non-abelian or not 2-generated, then the commuting graph is a spanning subgraph of the non-generating graph (the complement of the generating graph).

This graph has received a great deal of attention, partly in view of the fact that all non-abelian finite simple groups are 2-generated. Burness, Guralnick and Harper proved the following theorem, where we define the reduced generating graph by removing the identity:

Theorem

For a finite group G, the following three conditions are equivalent:

- *the reduced generating graph has no isolated vertices;*
- *any two vertices of the reduced generating graph have a common neighbour;*
- *every proper quotient of G is cyclic.*

Further graphs can be defined by putting different conditions on the group generated by x and y (for example, that they are nilpotent, or solvable). These have not been much studied.
I will conclude by defining another graph by a rather different procedure.

The deep commuting graph

We saw that the enhanced power graph of G is a spanning subgraph of the commuting graph of G. In the last part of my talk I will give a new construction, a graph (which I tentatively call the deep commuting graph of G, which lies between.

This graph is based on an idea suggested by Natalia Maslova. I have written it up with Bojan Kuzma; the preprint went on the arXiv today (`2012.03789`). We are grateful to Sean Eberhard, Saul Freedman, and Michael Giudici for helpful comments.

We require some preliminaries first.

Schur multiplier and Schur covers

Recall that, for a finite group G, a central extension of G is a group H with central subgroup Z such that $H/Z \cong G$; it is a stem extension if also $Z \leq H'$.
Schur showed that, for any finite group G, there is a largest abelian group Z such that there is a stem extension H of G with $H/Z \cong G$. The group $M(G) = Z$ is the Schur multiplier of G.
The corresponding group H is called a Schur cover of G.
Although $M(G)$ is unique, the Schur cover may not be. For example, the dihedral and quaternion groups of order 8 are both Schur covers of the Klein group.
$M(G)$ has several other descriptions: for example, it is $H^1(G, \mathbb{C}^\times)$; it is $(F' \cap R)/[R, F]$, where $G \cong F/R$ with F a free group (a presentation of G).

Definition of the graph

Let G be a finite group. Take a Schur cover H of G, with $H/Z \cong G$. We regard G as a quotient of H. For $x, y \in G$, join x to y if and only if their inverse images in H commute.
Note that the definition is independent of choice of the inverse image. For if a and b commute, then any element of the coset Za commutes with any element of Zb.
This definition appears to depend on the choice of Schur cover, but in fact it does not, as we will see.

Isoclinism

The commutation map γ maps $G/Z(G) \times G/Z(G)$ to G' by the rule $\gamma(Z(g)x, Z(g)y) = [x, y]$.
Two groups G_1, G_2 are isoclinic if there are isomorphisms $\phi : G_1/Z(G_1) \to G_2/Z(G_2)$ and $\psi : G_1' \to G_2'$ which take the commutation map of G_1 to the commutation map for G_2.

- If G_1 and G_2 are isoclinic groups of the same order, then their commuting graphs are isomorphic. (The isoclinism takes commuting pairs of cosets of $Z(G_1)$ in G_1 to the corresponding set in G_2.
- Any two Schur covers of a group G are isoclinic (a result of Jones and Wiegold).

So the deep commuting graph of G is independent of the choice of Schur cover, up to isomorphism. A closer look at the argument of Jones and Wiegold allows us to prove that it is unique: any two Schur covers give the same set of edges.

Properties

Theorem

Let Γ_1, Γ_2 and Γ_3 be the enhanced power graph, the deep commuting graph and the commuting graph of G.

- $E(\Gamma_1) \subseteq E(\Gamma_2) \subseteq E(\Gamma_3)$.
- $E(\Gamma_1) = E(\Gamma_2)$ *if and only if the following condition holds. Let H be a Schur cover of G, with $H/Z = G$. Take any subgroup A of G and B its inverse image in H, so that $Z \leqslant B$ and $B/Z = A$. If B is abelian, then A is cyclic.*

The proofs are straightforward. The relation between Γ_1 and Γ_2 depends on the fact that if $Z \leq Z(A)$ and A/Z is cyclic then A is abelian.

Commuting graph and deep commuting graph

The inclusion between these two graphs is more subtle.
A central extension H of G (with $G = H/Z$, $Z \leq Z(G)$) is said to be commutation-preserving, or CP for short, if whenever $x, y \in G$ and a, b are lifts of x, y in H, we have $[a, b] = 1$ if and only if $[x, y] = 1$. (The forward implication always holds.)
The Bogomolov multiplier $B_0(G)$ is the analogue of the Schur multiplier for CP extensions. It is the kernel of the largest CP stem extension of G. It is a quotient of the Schur multiplier. I refer to recent papers of Jezernik and Moravec for further information.

Theorem

The deep commuting graph of G is equal to the commuting graph of G if and only if $M(G) = B_0(G)$.

For failure of the CP property for x and y is equivalent to their being joined in the commuting graph but not in the deep commuting graph.

Examples

Using this theorem, we can find examples of groups G for which the enhanced power graph, deep commuting graph and commuting graph are all distinct.
Let G be the alternating group A_n with $n \geq 8$. The Schur multiplier of G is C_2, and there is a unique Schur cover.

- G has a subgroup $C_3 \times C_3$, which is non-cyclic, but its lift to the Schur cover is $C_2 \times C_3 \times C_3$, which is abelian. So the enhanced power graph is not equal to the deep commuting graph.
- The Schur cover is not a CP extension. For an involution in G lifts to an involution in the Schur cover if and only if the number of 2-cycles is divisible by 4; we can find two involutions with four 2-cycles whose product has two 2-cycles, so they commute but their lifts do not. So $B_0(G) = 1$ and $|M(G)| = 2$, so the deep commuting graph is not equal to the commuting graph.

To conclude

Kunyavskiĭ has shown that the Bogomolov multiplier of finite simple, quasisimple, or almost simple groups is trivial except for certain covers of $\mathrm{PSL}(3,4)$.
Bogomolov's work connected $B_0(G)$ with the work of Artin and Mumford on Noether's problem on the pure transcendence of the field of G-invariant functions on $\mathbb{C}(V)$.
There is surely more of interest to investigate here!

- Urban Jezernik and Primož Moravec, Commutativity preserving extensions of groups, *Proc. Royal Soc. Edinburgh*, Series A, **148** (2018), 575–592.

More on the GK graph

How many groups have a given graph as GK graph? The answer could be finite or infinite. But it is easy to see that there is a function F such that if more than $F(n)$ groups have a given n-vertex graph as GK graph, then infinitely many do. (Just take $F(n)$ to be the largest finite number of groups that occur for any n-vertex graph.)

How large is $F(n)$? Natalia Maslova and I have been looking at this question, and found that it is polynomially bounded.

Our present bound is $O(n^7)$, but we hope to improve this a bit ...