Graphs defined on groups

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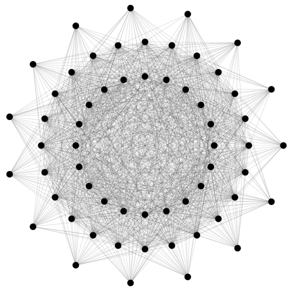

LTCC Intensive Course
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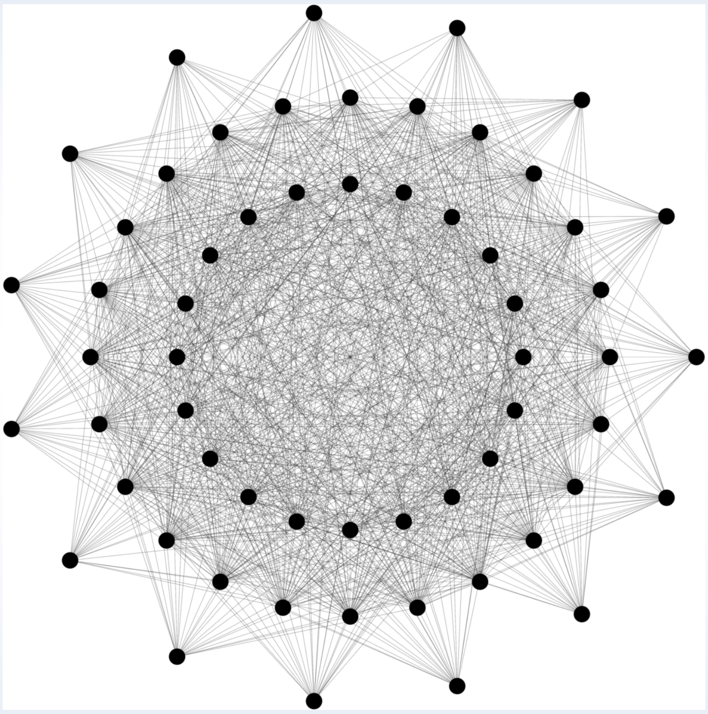
LTCC Online Intensive Course

For Researchers in the Mathematical Sciences

“Graphs defined on groups”

By Peter Cameron, University of St Andrews



The generating graph of the alternating group A_5

Image credit: Scott Harper, Bristol

The commuting graph, generating graph, power graph, and other graphs defined on a group carry important information about the group, as well as being useful for various applications. Moreover, they form a hierarchy, and studying them together can reveal many important connections.

Along the way, participants will meet many important properties of graphs (such as perfect graphs) and groups (such as finite simple groups), and see how information about graphs can throw light on groups. The subject includes many open problems, and the audience are encouraged to think about some of these.

Dates:

Tue 8 Jun (1pm to 5pm)
Wed 9 Jun (9am to 1pm)

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Lecture 1
Some of the players

Brauer and Fowler

In 1955, Brauer and Fowler published a paper which, in retrospect, was the first step on the thousand-mile journey to the Classification of Finite Simple Groups (CFSG).

- Richard Brauer and K. A. Fowler, On groups of even order, *Ann. Math.* **62** (1955), 565–583.

The paper is best known for the following theorem (though they do not state it explicitly as a theorem):

Theorem
Let H be a finite group. Then there are only finitely many finite simple groups G containing an involution z such that $C_G(z) \cong H$.

This immediately suggests the problem of characterising finite simple groups by the centraliser of an involution. This became a major constituent of CFSG.

A graph?

The paper of Brauer and Fowler does not contain the word "graph". However, it does contain the following definition. Let $G^\# = G \setminus \{1\}$. For $g, h \in G^\#$, the distance $d(g, h)$ is the smallest d for which there exist $g_0, g_1, \ldots, g_d \in G^\#$ such that $g_0 = g$, $g_d = h$, and $g_{i-1}g_i = g_ig_{i-1}$ for $i = 1, 2, \ldots, d$.
This is obviously the distance in the graph with vertex set $G^\#$, in which two vertices are joined by an edge if they commute. This was the first appearance of what is now known as the reduced commuting graph of G.

Here is a simple example of their argument.

Proposition
Let x and y be non-conjugate involutions in a group G. Then $d(x, y) \leq 2$.

Proof.
The subgroup generated by x and y is a dihedral group D_{2n} of order $2n$. Now n must be even, since if it were odd then x and y would generate Sylow subgroups of $\langle x, y\rangle$, and so would be conjugate, contrary to hypothesis. So D_{2n} contains a central involution z, which commutes with both x and y. □

A simple enough argument, but it shows the blend of group theory and graph theory you should expect in the remainder of this course.

Where are we going?

We have seen the first tentative appearance of the commuting graph of a finite group.
There are a number of further graphs that have been considered; some of these already have a large literature. These include the power graph, enhanced power graph, deep commuting graph, generating graph, nilpotency graph, solubility graph, and Engel graph.
My interest will be not so much in the individual graphs, as in the relations between them. With a little twist, these graphs form a hierarchy on a given group, with each one contained in the next. This will focus our attention on two things: common properties of the graphs, and how they relate; and properties of further graphs which are formed by differences of the edge sets of two graphs in the hierarchy.

Aims

I hope in this course to

- introduce you to an area of algebraic graph theory which I find fascinating and addictive;
- mention a number of open problems;
- give you some tools to tackle them.

For more details, see my paper "Graphs defined on groups", to appear in the *International Journal of Group Theory*: the doi is

`10.22108/ijgt.2021.127679.1681`

or (better) get it from

`https://ijgt.ui.ac.ir/article_25608.html`

or you can find a version on the arXiv, `2102.11177`.

Thanks ...

- to Ambat Vijayakumar, who invited me to present some of this material in a couple of seminars in a new series he set up in Kochi in Kerala, India;
- to Alireza Abdollahi, who saw the preprint and encouraged me to publish it in the Journal he edits;
- to Scott Harper, who drew the beatiful picture of the generating graph of A_5, for permission to use it on the poster and title pages;
- to many coauthors, several of whom will be mentioned in what follows, for collaborations;
- and, of course, to the London Taught Course Centre for the opportunity to preach about them here.

Where we are not going

I will not be talking, except in passing, about Cayley graphs. These are graphs defined on groups, and have a huge theory; much of algebraic graph theory, and arguably most of geometric group theory, concerns Cayley graphs (of finite and infinite groups respectively).
To recall: if S is an inverse-closed subset of $G \setminus \{1\}$, the Cayley graph $\mathrm{Cay}(G,S)$ has vertex set G, with an edge from x to y if $xy^{-1} \in S$. (It is slightly different if you like left actions.)
A Cayley graph is a graph whose vertex set is a group G and which is invariant under right translations by elements of G. It is not invariant under automorphisms of G except in very special cases. By contrast, the graphs I am discussing are invariant under automorphisms of G, because they are uniquely specified by G, without requiring choosing a generating set.

Notation

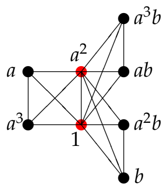
A talk about groups typically begins "Let G be a group ...", while a talk about graphs will start "Let G be a graph ...". We will be talking about both, so we have to make a decision.

- "Graph" is a Greek word, so it makes sense for a graph to be Γ.
- "Group" is a German word, so perhaps a group should be $\mathfrak{G}$; but I never learned how to do a Gothic G in handwriting, and probably you didn't either, so I will use G for a group.

Otherwise, notation for groups and graphs will be standard. I will try to explain as I go along, but please ask if you need clarification!

Dramatis Personae, 1: the commuting graph

The commuting graph $\mathrm{Com}(G)$ of G has vertex set G; vertices g and h are joined if and only if $gh = hg$. (This definition would put a loop at every vertex; we silently suppress these.)
Here are the commuting graphs of the two non-abelian groups of order 8: $D_8 = \langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$.



The reduced commuting graph

There are two conventions we need to consider.

- The formal definition would require each vertex to be joined to itself; that is, the graph has a loop at every vertex. We will see soon that there is sometimes a good reason for this. But usually we will silently remove the loops.
- You will recall that Brauer and Fowler removed the identity from the graph; the identity commutes with everything and so is joined to all vertices, thus questions like connectedness (which was important for them) would become trivial. My default is that all graphs are defined on the whole group; when we come to consider connectedness, we first determine which vertices are joined to all others, and then remove them.

We will denote the commuting graph of G (defined on all of G, but without loops) by $\mathrm{Com}(G)$.

The "Burnside process"

Another application of the commuting graph comes from a completely different area.

Theorem (Orbit-counting Lemma)

Let G be a permutation group on a finite set Ω. Then the number of orbits of G on Ω is equal to the average number of fixed points of elements of G.

The proof involves constructing a bipartite graph whose vertex set is $G \cup \Omega$, with an edge from $g \in G$ to $x \in \Omega$ if g fixes x. Now counting the number of edges in the graph in two different ways gives the result.

Mark Jerrum showed that more is true. Consider the uniform random walk on the graph just constructed: at each time step we move from a vertex to a neighbouring vertex chosen uniformly at random.

A small adaptation of the proof of the Orbit-counting Lemma shows that, if we start at a vertex in Ω and take an even number of steps (so that we are back in Ω), the limiting distribution is uniform on orbits – that is, the probability of being at a point $x \in \Omega$ is inversely proportional to the size of the orbit containing x.

Jerrum called this random walk the Burnside process, since the Orbit-counting Lemma was referred to (incorrectly) by early combinatorial enumerators as "Burnside's Lemma" (it appears without attribution in the second edition of Burnside's book). Peter Neumann traced it back to Cauchy and Frobenius.

Conjugacy classes

A group G acts on itself by conjugation. In this case $\Omega = G$, so we can identify these two sets. Now the group element g fixes x if and only if $gx = xg$. So, for this action, the Burnside process is just a random walk on the commuting graph of G (including the identity, and with a loop at each vertex).

The importance of this is that some very large groups have very small conjugacy classes. For an extreme example, the symmetric group S_n has order $n!$, but the transpositions form a conjugacy class of size just $n(n-1)/2$. If we are trying to find all conjugacy classes in a large group, the random walk "magnifies" such small classes and makes them more visible.

Persi Diaconis has used similar ideas to show that the problem of describing conjugacy classes in high-dimensional analogues of Heisenberg groups over finite fields is likely to be hard, since their commuting graphs are arbitrarily complicated.

Dramatis Personae, 2: The power graph and its relatives

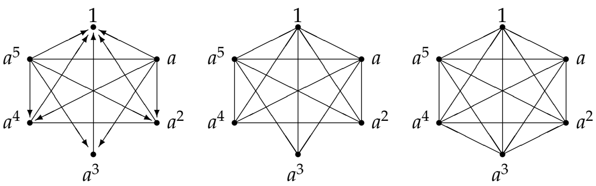
The power graph of a group G was first defined by Kelarev and Quinn as a directed graph, with an arc $x \to y$ from x to y whenever y is a power of x. We denote this directed graph by $\mathrm{DPow}(G)$.

Chakrabarty, Ghosh and Sen introduced the undirected version $\mathrm{Pow}(G)$, in which x and y are joined if $x \to y$ or $y \to x$ (or both) in the directed power graph.

Although pre-empted by Abdollahi, Aalipour *et al.* introduced the enhanced power graph $\mathrm{EPow}(G)$, in which x and y are joined if there exists an element z such that $z \to x$ and $z \to y$ in the directed power graph.

Note that the edge set of the power graph is contained in that of the enhanced power graph (hence the name).

An example: C_6



The pictures show the directed power graph, the power graph, and the enhanced power graph of the cyclic group C_6.

In the directed power graph, if two elements generate the same cyclic group, then there are arcs in both directions: we represent this by an undirected edge. To get the power graph, we simply ignore the remaining edges.

Note that, in $\mathrm{Pow}(C_6)$, we cannot distinguish between the identity and the two generators; each is joined to all other vertices.

Relations

Theorem

For two groups G and H, the following are equivalent:

1. $\mathrm{DPow}(G) \cong \mathrm{DPow}(H)$;
2. $\mathrm{Pow}(G) \cong \mathrm{Pow}(H)$;
3. $\mathrm{EPow}(G) \cong \mathrm{EPow}(H)$.

The implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ come from the definitions. $2 \Rightarrow 1$ was proved by Cameron, and $3 \Rightarrow 1$ by Zahirović.

The implication $2 \Rightarrow 1$ does not imply that the directed power graph can be recovered uniquely from the power graph. As we have seen, in the power graph of C_6, the identity and the two generators are indistinguishable, whereas one is a sink and the other two sources in the directed power graph. In $\mathrm{EPow}(C_6)$, all vertices are indistinguishable.

Also, the three equivalent implications do not imply that $G \cong H$: any two groups of exponent 3 with the same order have isomorphic power graphs (consisting of a number of triangles with a common vertex).
However, this does hold in a special case. We cannot distinguish abelian groups of the same order by their commuting graphs, but we can by their power graphs:

Theorem
If G and H are abelian groups with $\mathrm{Pow}(G) \cong \mathrm{Pow}(H)$, then $G \cong H$.

Proof.
Cameron and Ghosh showed that, from the power graph of G, we can reconstruct the numbers of elements of each possible order in G. For abelian groups, this data determines the group up to isomorphism. □

Another view

Proposition
In the group G,

- *x and y are joined in the commuting graph if and only if $\langle x, y \rangle$ is abelian.*
- *x and y are joined in the enhanced power graph if and only if $\langle x, y \rangle$ is cyclic.*

This suggests an obvious generalisation: choose your favourite family of groups, and join x to y if and only if $\langle x, y \rangle$ belongs to that family.
In particular, x and y are joined in the nilpotency graph of G if $\langle x, y \rangle$ is nilpotent; and are joined in the solubility graph of G if $\langle x, y \rangle$ is soluble.
More on these later.

The generating graph

Instead, we follow a different take on this idea. The generating graph $\mathrm{Gen}(G)$ of G has vertex set G, with vertices x, y joined if $\langle x, y \rangle = G$. Clearly it is a null graph if G cannot be generated by two elements; but we know from CFSG that all finite simple groups can be generated by two elements, so there are interesting examples to consider.
The generating graph for many interesting groups is fairly dense, as the following result of Burness, Guralnick and Harper shows. We say that a graph has spread k if any k vertices have a common neighbour. Thus, "spread 1" means "no isolated vertices, while "spread 2" means that any two vertices are joined by a path of length 2 (so the diameter is at most 2).

The reduced generating graph is the generating graph with the identity removed.

Theorem
For a finite group G, the following are equivalent:

- $\mathrm{Gen}(G)$ *has spread* 1;
- $\mathrm{Gen}(G)$ *has spread* 2;
- *every proper quotient of G is cyclic.*

So for example every non-abelian finite simple group satisfies these conditions.
For reasons which will become clear, I will talk about the non-generating graph $\mathrm{NGen}(G)$, the complement of the generating graph. This also turns out to be connected with small diameter for non-abelian simple G (if the identity is removed).

The deep commuting graph

This graph is a bit different from the others, requiring more serious group theory for its definition (by Cameron and Kuzma).
Let G be a finite group. A central extension of G is a group H with a normal subgroup Z contained in the centre of H such that $H/Z \cong G$. We regard the epimorphism from H to G as part of the structure of the extension, and call Z the kernel. So we can talk about the inverse images of an element of G in H.
Now the deep commuting graph $\mathrm{DCom}(G)$ of G is the graph with vertex set G, in which x and y are joined if and only if their inverse images in every central extension of G commute.
So it is not obvious that the definition makes sense. But we will see that it is enough to consider one central extension.

Schur covers and Schur multiplier

A central extension H of G with kernel Z is a stem extension of G if $Z \leq Z(H) \cap H'$, where H' is the derived group or commutator subgroup of H.
Schur showed the following:

Theorem
Let G be a finite group. Then there is a stem extension H of G which is of maximal order. Moreover, in any two stem extensions of maximal order, the kernels are isomorphic.

The stem extensions of maximal order are called Schur covers of G, and the kernel is the Schur multiplier of G.

The Schur multiplier

The Schur multiplier occurs in many other disguises. For example:

- It is the second homology group of G over the integers, $H_2(G, \mathbb{Z})$.
- It is the second cohomology from group of G over the multiplicative group of complex numbers, $H^2(G, \mathbb{C}^\times)$.
- If we have a presentation of G as F/R, where F is a free group, then the Schur multiplier is $(R \cap F')/[R, F]$.

The deep commuting graph

Theorem

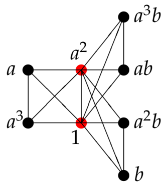
Let H be a Schur cover of G. Then two elements of G have the property that their inverse images in every central extension of G commute if and only if their inverse images in H commute.

Thus the deep commuting graph of G is well-defined; it is obtained by taking the commuting graph of a Schur cover of G and projecting it onto G.

As a corollary we see that any two Schur covers of G have isomorphic commuting graphs. This can be proved directly using the notion of isoclinism.

An example

The Klein group $V_4 = C_2 \times C_2$ has two Schur covers, the dihedral and quaternion groups of order 8 (so that its Schur multiplier is C_2). Here is the commuting graph of these groups again:



We see that the deep commuting graph of V_4 is the star $K_{1,3}$, even though its commuting graph is the complete graph K_4.

Invariance under automorphisms

For most of the graph types we have defined (the power graph, enhanced power graph, commuting graph, and generating graph), it is clear that any automorphism of the group G induced an automorphism of the corresponding graph on G. This is not immediately clear for the deep commuting graph, but it is true in this case. Once we know that our original definition (two vertices joined if their inverse images in every central extension commute) is a good definition, it is clear that the graph is preserved by automorphisms.

Lecture 2
The hierarchy

The hierarchy of graphs

The joining rules for elements x and y of a group G:

- null graph: never
- power graph $\mathrm{Pow}(G)$: one is a power of the other
- enhanced power graph $\mathrm{EPow}(G)$: both are powers of an element z; equivalently $\langle x, y\rangle$ is cyclic
- deep commuting graph $\mathrm{DCom}(G)$: the inverse images of x and y commute in every central extension of G
- commuting graph $\mathrm{Com}(G)$: $xy = yx$; equivalently $\langle x, y\rangle$ is abelian
- non-generating graph $\mathrm{NGen}(G)$: $\langle x, y\rangle \neq G$
- complete graph: always

I will call this the graph hierarchy of G.

Inclusions

Proposition

With the possible exception of $\mathrm{Com}(G)$ and $\mathrm{NGen}(G)$, the edge set of each graph in the hierarchy is contained in that of the next. This holds for $\mathrm{Com}(G)$ and $\mathrm{NGen}(G)$ if and only if G is either non-abelian or not 2-generated.

Proof.

Everything is clear except perhaps the position of the deep commuting graph. It is clear that edges of the deep commuting graph are edges of the commuting graph, since G is a central extension of itself.

Suppose that x and y are joined in the enhanced power graph, so that $\langle x, y\rangle = \langle z\rangle$ for some z. Let H be a central extension of G, with $H/Z \cong G$, and let a, b, c be inverse images of x, y, z in H. Then $\langle Z, a, b\rangle = \langle Z, c\rangle$ is abelian, since Z is central; so a and b commute. $\square$

When does equality hold?

A natural question that arises is: for which groups G is a consecutive pair of graphs in the hierarchy on G equal?

At the top and bottom, this is easy:

- $\mathrm{Pow}(G)$ is null if and only if $G = \{1\}$. For every non-trivial element is joined to 1 in the power graph.
- $\mathrm{NGen}(G)$ is complete if and only if G is not 2-generated.
- $\mathrm{Com}(G) = \mathrm{NGen}(G)$ if and only if G is a minimal non-abelian group.

The minimal non-abelian groups were determined by Miller and Moreno in 1904. (I give their result on the next slide.) They are all 2-generated. So, if G is not minimal non-abelian, then it contains two non-commuting elements which generate a proper subgroup.

For the other gaps, I ignore the deep commuting graph at first, and come back to it later.

Minimal non-abelian groups

Let G be a minimal non-abelian group. There are two possibilities:

- $|G|$ is a power of a prime p, and $G = \langle a, b\rangle$, where $Z(G) = \langle a^p, b^p, [a, b]\rangle$, and $G' = \langle [a, b]\rangle$ has order p. Moreover, either $p = 2$ and G is the quaternion group of order 8, or $\langle a^p\rangle \cap \langle b^p\rangle = \{1\}$.
- $|G|$ is divisible by two primes p and q; moreover, G is a semidirect product of an elementary abelian p-group N by a cyclic q-group $\langle b\rangle$, where b induces an automorphism of order q which is irreducible on N.

As noted, it is important for us that minimal non-abelian groups are 2-generated.

Frobenius and 2-Frobenius groups

The group G is a Frobenius group if it has a proper subgroup H (called a Frobenius complement) with the property that $H \cap H^g = \{1\}$ for all $g \in G \setminus H$. The symmetric group S_3 is an example.

Frobenius showed that, if N is the set of elements lying in no conjugate of H, together with the identity, then N is a normal subgroup of G, called the Frobenius kernel. Moreover, Thompson showed that the Frobenius kernel is nilpotent, and Zassenhaus determined the structures of Frobenius complements.

The group G is a 2-Frobenius group if it has a chain of normal subgroups $\{1\} \lhd N \lhd M \lhd G$ such that

- M is a Frobenius group with Frobenius kernel N;
- G/N is a Frobenius group with Frobenius kernel M/N.

The symmetric group S_4 is an example.

Power graph and enhanced power graph

Proposition

Let G be a finite group. Then the following are equivalent:

- $\mathrm{Pow}(G) = \mathrm{EPow}(G)$;
- *G contains no subgroup $C_p \times C_q$ for distinct primes p, q;*
- *every element of G has prime power order.*

Proof.

Commuting elements of distinct prime orders p and q are joined in the enhanced power graph but not in the power graph. Conversely, if x and y are joined in the enhanced power graph but not in the power graph, then $\langle x, y \rangle$ is cyclic but not of prime power order, so it contains an element of order pq for distinct primes p, q. □

Groups with the last property are called EPPO groups.

Classification of EPPO groups

If G is an EPPO group, then the centraliser of an involution in G must be a 2-group. Back in the last century, groups with this property were studied under the name CIT groups by Suzuki, who classified the simple CIT groups:

- $\mathrm{PSL}(2, q)$ for q a power of 2;
- the Suzuki group $\mathrm{Sz}(q)$, for q an odd power of 2;
- $\mathrm{PSL}(2, q)$, where q is a Fermat or Mersenne prime or $q = 9$;
- $\mathrm{PSL}(3, 4)$.

The non-simple case took longer; following the work of a number of mathematicians, Natalia Maslova and I completed the classification of EPPO groups in a paper now on the arXiv. This is given on the next slide, where $\pi(G)$ denotes the set of prime divisors of $|G|$.

Theorem

An EPPO group G satisfies one of the following:

- *$|\pi(G)| = 1$ and G is a p-group.*
- *$|\pi(G)| = 2$ and G is a solvable Frobenius or 2-Frobenius group.*
- *$|\pi(G)| = 3$ and $G \in \{A_6, \mathrm{PSL}_2(7), \mathrm{PSL}_2(17), M_{10}\}$.*
- *$|\pi(G)| = 3$, $G/O_2(G)$ is $\mathrm{PSL}_2(2^n)$ for $n \in \{2, 3\}$ and if $O_2(G) \neq \{1\}$, then $O_2(G)$ is the direct product of minimal normal subgroups of G, each of which is of order 2^{2n} and as a $G/O_2(G)$-module is isomorphic to the natural $\mathrm{GF}(2^n)SL(2^n)$-module.*
- *$|\pi(G)| = 4$ and $G \cong \mathrm{PSL}_3(4)$.*
- *$|\pi(G)| = 4$, $G/O_2(G)$ is $\mathrm{Sz}(2^n)$ for $n \in \{3, 5\}$, and if $O_2(G) \neq \{1\}$, then $O_2(G)$ is the direct product of minimal normal subgroups of G, each of which is of order 2^{4n} and as a $G/O_2(G)$-module is isomorphic to the natural $\mathrm{GF}(2^n)\,\mathrm{Sz}(2^n)$-module of dimension 4.*

The Gruenberg–Kegel graph appears

The Gruenberg–Kegel graph of a finite group G is the graph whose vertex set is $\pi(G)$, with an edge from p to q if and only if G contains an element of order pq.

I will have a lot more to say about this graph later in the course, but for now let us just note the following:

Proposition

The finite group G is an EPPO group if and only if its Gruenberg–Kegel graph has no edges.

An important ingredient of the classification of EPPO groups is the unpublished theorem of Gruenberg and Kegel, a structure theorem about groups whose Gruenberg–Kegel graph is not connected. More on this later.

Enhanced power graph and commuting graph

Proposition

Let G be a finite group. Then the following are equivalent:

- $\mathrm{EPow}(G) = \mathrm{Com}(G)$;
- *G contains no subgroup $C_p \times C_p$ for prime p;*
- *the Sylow subgroups of G are cyclic or generalised quaternion groups.*

Proof.

The equivalence of the first two conditions is clear since a non-cyclic abelian group contains a subgroup $C_p \times C_p$.

The equivalence of the second and third follows from a theorem of Burnside asserting that groups of p-power order containing no $C_p \times C_p$ subgroup must be cyclic or generalised quaternion. □

Classification

It is possible to determine completely the groups with cyclic or generalised quaternion Sylow subgroups.

If all Sylow subgroups are cyclic, then G is metacyclic. For let $F(G)$ be the Fitting subgroup of G, the largest normal nilpotent subgroup of G. Then $F(G)$ is a direct product of cyclic groups of coprime orders, so is cyclic. Since it contains its centraliser, $G/F(G)$ embeds into $\mathrm{Aut}(F(G))$, which is abelian with cyclic Sylow subgroups and so is cyclic.

If p and q are primes such that $q \mid p - 1$, the non-abelian group of order pq (the semidirect product of C_p by C_q) is an example.

Now suppose that the Sylow 2-subgroups of G are generalised quaternion and the odd Sylow subgroups cyclic.
Let $O(G)$ be the largest normal subgroup of G of odd order.
Then by the previous analysis, $O(G)$ is metacyclic. Put $\bar{G} = G/O(G)$.
By Glauberman's Z^*-theorem, $\bar{G}$ has a unique central subgroup of order 2, generated by z say. Then $\bar{G}/\langle z\rangle$ has dihedral Sylow 2-subgroups, and so is determined by the Gorenstein–Walter theorem; it must be $\mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$, for an odd prime p.
There is a unique group $\bar{G}$ for each choice of $\bar{G}/\langle z\rangle$. Indeed, a cohomological argument due to Glauberman shows that any group with dihedral Sylow 2-subgroups has a unique extension containing only one involution (and so having generalised quaternion Sylow 2-subgroups).

The deep commuting graph

We deferred discussion of this graph, which lies between the enhanced power graph and the commuting graph in the hierarchy. For which groups is it equal to one or other of these?

Theorem
Let G be a finite group. Then $\mathrm{DCom}(G) = \mathrm{EPow}(G)$ if and only if G has the following property: let H be a Schur cover of G, with $H/Z = G$. Then for any subgroup A of G, with B the corresponding subgroup of H (so $Z \leqslant B$ and $B/Z = A$), if B is abelian, then A is cyclic.

Proof.
Just a matter of checking the definitions: $\langle x,y\rangle$ is cyclic if x and y are joined in $\mathrm{EPow}(G)$, and their inverse images in a Schur cover generate an abelian group if and only if x and y are joined in $\mathrm{DCom}(G)$. □

An example

Suppose that G is a group and p a prime such that G has a subgroup $H \cong C_p \times C_p$, and p does not divide the order of the Schur multiplier $M(G)$. Then the lift of H to a Schur cover splits over the Schur multiplier, and hence is abelian; but H is not cyclic. So $\mathrm{DCom}(G) \neq \mathrm{EPow}(G)$.
For example, the Schur multiplier of the alternating group A_n for $n \geq 8$ is cyclic of order 2, and this group contains $C_3 \times C_3$.

The Bogomolov multiplier

For the other equality, we need another piece of technology. Recall that

- a stem extension of G is a group H with a subgroup $Z \leq Z(H) \cap H'$ such that $H/Z \cong G$;
- a Schur cover is a largest stem extension, and the Schur multiplier of G is the subgroup Z (it is determined uniquely by G, although H may not be).

The Schur multiplier of G is denoted by $M(G)$.
Let H be a stem extension of G, and let a, b be the inverse images in H of $x, y \in G$. Clearly, if a and b commute, then so do x and y. If the converse is true, then we say that the extension is commutativity-preserving, or CP for short.

Theorem
Given a finite group G, there is a unique finite abelian group Z such that any CP stem extension of G of largest possible order has kernel Z.

The subgroup Z is called the Bogomolov multiplier of G, denoted by $B_0(G)$.
It has various descriptions. For example, one can define a nonabelian exterior square $G \wedge G$, generated by symbols $x \wedge y$ for $x, y \in G$ subject to the relations

$$(xy)\wedge z = (x^y \wedge z^y)(y \wedge z), \quad x \wedge (yz) = (x \wedge z)(x^z \wedge y^z), \quad x \wedge x = 1.$$

Then $x \wedge y \mapsto [x,y]$ is a surjective homomorphism from $G \wedge G$ to G' whose kernel is $M(G)$. If we set $M_0(G) = \langle x \wedge y \mid [x,y] = 1\rangle$; then $B_0(G) \cong M(G)/M_0(G)$.

The Bogomolov multiplier arose in connection with the work of Artin and Mumford on obstructions to Noether's conjecture on the pure transcendence of the field of invariants. Fortunately we do not need this background.
More practially, the GAP package `HAP`, written by Graham Ellis, will compute the Bogomolov multiplier, as well as the Schur multiplier, of a group.

Theorem
Let G be a finite group. Then $\mathrm{DCom}(G) = \mathrm{Com}(G)$ if and only if $B_0(G) = M(G)$.

Simple groups

Kunyavskiĭ proved a conjecture of Bogomolov by showing that the Bogomolov multiplier of every finite non-abelian simple group is trivial.
The alternating group A_n for $n \geq 8$ has Schur multiplier of order 2. So for these groups, the enhanced power graph, deep commuting graph, and commuting graph are all unequal.
However, the Schur multiplier of M_{11} is trivial; so the commuting graph and deep commuting graph of this group are equal.

Further examples

It is possible for the Schur and Bogomolov multipliers to be equal when they are both non-trivial. An example is a certain group of order 64 (this is `SmallGroup(64,182)` in the GAP library).
Dihedral groups of order $2^n \geq 8$ have the property that their deep commuting graphs are equal to their enhanced power graphs, but not equal to their commuting graphs.
It has to be admitted that the situation is not perfectly understood ...

Differences

Now that we have some kind of description of groups for which two graphs in the hierarchy coincide, a natural question is: if G is a group for which two of these graphs are unequal, what can be said about the graph whose edge set is the difference of the edge sets of the two graphs?
Not very much is known. There are a couple of trivial observations:

- The difference between $\mathrm{Pow}(G)$ and the null graph is just $\mathrm{Pow}(G)$, which has an extensive literature.
- The difference between the complete graph and the non-generating graph is, of course, the generating graph $\mathrm{Gen}(G)$, which also has an extensive literature.

But there are many more differences that could be explored. Some of these will arise later in this course.

The non-commuting, non-generating graph

One of these difference graphs which has been studied in some detail is the difference between the non-generating graph and the commuting graph (which, we recall, is non-null if and only if G is not a minimal non-abelian group).
This is considered by Saul Freedman in his PhD thesis, currently nearing completion. He concentrates mainly on questions of connectedness and of diameter of connected components.
Since this material is not yet available, I will not discuss it here.

An observation

The power graph, enhanced power graph, deep commuting graph, and commuting graph have the following property:

> Let χ be one of the above graph types, and G a finite group. If H is a subgroup of G, then the induced subgraph of $\chi(G)$ on H is $\chi(H)$.

But the non-generating graph behaves very differently:

> Let G be a finite group. If H is a proper subgraph of G, then the induced subgraph of $\mathrm{NGen}(G)$ on H is a complete graph.

For no two elements of H can generate G.

Lecture 3
Cographs and twin reduction

Graph theory definitions

I have used some of these ideas informally already: here are definitions.
Our graphs are always simple (without loops and multiple edges).
A walk from v to w is a sequence $(v_0, v_1, \ldots, v_r)$ of vertices such that $v_0 = v$, $v_r = w$, and v_{i-1} is joined to v_i for $i = 1, \ldots, r$. It is a path if the sequence has no repeated vertices. (If there is a walk from v to w then there is a path.) A graph is connected if there is a path between any two of its vertices. In a connected graph, the distance from v to w is the length (one less than the number of vertices) of the smallest path joining them, and the diameter of the graph is the maximum distance between two vertices.
The complement of a graph Γ is the graph Γ^c with the same vertex set whose edges are those pairs of vertices which are not edges in Γ.

Subgraphs

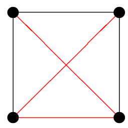
Let Γ be a graph. We denote its vertex set by $V(\Gamma)$ and its edge set by $E(\Gamma)$.
A subgraph of Γ has as vertex and edge sets subsets of those of Γ, with the proviso that if an edge belongs to the subgraph then so do both of its vertices.
Two kinds of subgraphs are particularly important:

- For an induced subgraph, we take a subset W of $V(\Gamma)$ as vertex set, and *all* edges of Γ with both vertices in W as the edge set.
- For a spanning subgraph, the vertex set is *all* of $V(\Gamma)$, and the edge set is a subset of $E(\Gamma)$.

Note that, in our graph hierarchy on a group G, each graph is a spanning subgraph of the next.

Cographs

Cographs form a class of graphs with many nice properties. They have an inductive structure which allows many hard algorithmic problems to be solved very easily on cographs.
To motivate the definition, note that the complement of a disconnected graph is connected. For if Γ is disconnected, then the vertex set can be split into two non-empty parts A and B with no edges between them. Now in Γ^c, every vertex of A is joined to every vertex of B; so any two vertices in A have a common neighbour in B, and *vice versa*.
The converse is false, as the graph P_4 (the four-vertex path) shows:



Cographs

The next theorem describes graphs for which the converse of the above holds "inductively".

Theorem
For a finite graph Γ, the following conditions are equivalent:

1. *Γ has no induced subgraph isomorphic to P_4;*
2. *for every induced subgraph Δ of Γ with more than one vertex, either Δ or its complement Δ^c is disconnected;*
3. *Γ can be built from the trivial graph by the operations "disjoint union" and "complement".*

A graph satisfying these three conditions is called a cograph.

Proof
$1 \Rightarrow 2$: Suppose that Γ has no induced P_4 but both Γ and its complement are connected, and let Γ be minimal with this property. Then, given any vertex v, if we remove v (that is, take the induced subgraph on $V(\Gamma) \setminus \{v\}$), either the graph or its complement is disconnected, without loss the former.
I claim that v is joined to all other vertices of Γ. For we can partition $V(\Gamma)$ into two parts A and B so that every path between them passes through Γ. If some vertex u of A were not joined to v, we could take a path of length at least 2 from u to v and an edge from v to a vertex of B, giving an induced path of length 3, contrary to assumption. Similarly for B.
But now v is an isolated vertex in Γ^c, which is therefore disconnected.

$2 \Rightarrow 3$: By repeatedly splitting into connected components and taking the complement, a graph satisfying 2 is reduced to 1-vertex graphs. Reversing the splitting procedure gives the required construction.

$3 \Rightarrow 1$: It is clear that P_4 cannot be built in this way: if a graph contains P_4, then so does its complement; and if a graph contains P_4, then at least one of its connected components does. □

We see that cographs form the smallest class of graphs containing the 1-vertex graph and closed under complement and disjoint union.

Cographs have been rediscovered a number of times, and have received several different names in the literature, such as "complement-reducible graphs", "hereditary Dacey graphs", and "N-free graphs".
Here is some data. Almost all groups of order up to n are 2-groups. The table gives the number of groups, and the number for whom a graph in the hierarchy is a cograph.

Order	Groups	Pow	EPow	Com	NGen
1	1	1	1	1	1
2	1	1	1	1	1
4	2	2	2	2	2
8	5	5	5	5	5
16	14	14	14	14	14
32	51	51	51	44	51
64	267	267	267	152	267
128	2328	2328	2328	789	2328

And here are some results for small non-abelian simple groups:

G	$\lvert G\rvert$	Pow	EPow	DCom	Com	NGen
A_5	60	Y	Y	Y	Y	N
$\mathrm{PSL}(2,7)$	168	Y	Y	Y	N	N
A_6	360	Y	Y	Y	N	N
$\mathrm{PSL}(2,8)$	504	Y	Y	Y	Y	N
$\mathrm{PSL}(2,11)$	660	Y	Y	Y	N	N
$\mathrm{PSL}(2,13)$	1092	Y	Y	Y	N	N
$\mathrm{PSL}(2,17)$	2448	Y	Y	Y	N	N
A_7	2520	N	N	N	N	N
$\mathrm{PSL}(2,19)$	3420	Y	Y	Y	N	N
$\mathrm{PSL}(2,16)$	4080	Y	Y	Y	Y	N
$\mathrm{PSL}(3,3)$	5616	N	N	N	N	N
$\mathrm{PSU}(3,3)$	6048	N	N	N	N	N
$\mathrm{PSL}(2,23)$	6072	N	Y	Y	N	N
$\mathrm{PSL}(2,25)$	7800	N	Y	Y	N	N
M_{11}	7920	N	N	N	N	N

Some explanations

We have seen that, for groups of prime power order, the power graph and enhanced power graph are equal; we will see later that the power graph is a cograph.

Theorem

- *If G has prime power order, then $\mathrm{NGen}(G)$ is a cograph.*
- *If G is a non-abelian finite simple group, then $\mathrm{NGen}(G)$ is not a cograph.*

Proof.
For the first, if G is not 2-generated, then $\mathrm{NGen}(G)$ is complete; if it is 2-generated, then by the Burnside Basis Theorem, any subgroup of index p induces a complete graph, and any two of these complete graphs intersect in the Frattini subgroup $\Phi(G)$ (with index p^2); all other pairs generate.
For the second, we will see later that the generating graph of a finite simple group and its complement are both connected. □

The power graph of a p-group is a cograph

Cameron, Manna and Mehatari showed something a bit stronger: the power graph of a p-group has no induced P_4 or C_4.
Suppose first that (x,y,z) is an induced P_3. In $\mathrm{DPow}(G)$, we cannot have $x \to y \to z$ or $z \to y \to x$, since either would imply $x \sim z$ in $\mathrm{Pow}(G)$. Also we cannot have $y \to x$ and $y \to z$, since then $x,z \in \langle y\rangle$, but the power graph of a cyclic p-group is a complete graph. So we must have $x \to y$ and $z \to y$.
Now suppose that (x,y,z,w) is a path of length 4, with $x \not\sim z$ and $y \not\sim w$. Then we have $x \to y$ and $z \to y$, and also $y \to z$ and $w \to z$; but these imply $x \to z$, a contradiction. So both induced P_4 and induced C_4 are excluded.

The power graph of a nilpotent group

In the same paper, the following theorem is proved:

Theorem
Let G be a nilpotent group whose power graph is a cograph. Then either G is a p-group for some prime p, or G is cyclic of order pq, where p and q are distinct primes.

This theorem is more useful than it appears, since it restricts the possible nilpotent subgroups of an arbitrary group whose power graph is a cograph.
We will examine the groups $\mathrm{PSL}(2,q)$ on the next slide. Here q is a prime power. If q is a power of 2, let $\{l,m\} = \{q-1,q+1\}$; if q is odd, let $\{l,m\} = \{(q-1)/2,(q+1)/2\}$. Note that $\mathrm{PSL}(2,q)$ has maximal cyclic subgroups of orders l and m.

Proposition

With the notation just introduced, $\mathrm{Pow}(\mathrm{PSL}(2, q))$ *is a cograph if and only if each of l and m is either a prime power or the product of two distinct primes.*

Deciding which prime powers have this property is a number-theoretic property, and probably rather a hard one. The numbers $d \leq 200$ for which $q = 2^d$ has the above property are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199. For example, $2^{11} - 1 = 23 \cdot 89$, while $2^{11} + 1 = 3 \cdot 683$. The odd prime powers up to 500 with the property are 3, 5, 7, 9, 11, 13, 17, 19, 27, 29, 31, 43, 53, 67, 163, 173, 243, 257, 283, 317.

Question

Are there infinitely many prime powers q for which the power graph of $\mathrm{PSL}(2, q)$ *is a cograph?*

Twins

Two vertices v and w in a graph Γ are called twins if they have the same neighbourhood (except possibly for one another). If we denote by $N(v)$ the set of vertices joined to v, and $\bar{N}(v) = \{v\} \cup N(v)$, then we call v and w open twins if $N(v) = N(w)$, and closed twins if $\bar{N}(v) = \bar{N}(w)$. A vertex cannot have both a closed and an open twin. For suppose that u and v are closed twins, and v and w are open twins. Then u and w are not joined (since u's twin v is not joined to w) and also joined (since w's twin v is joined to u), a contradiction.
Thus, being twins in a graph is an equivalence relation.
Note that interchanging twins (while fixing all other vertices) is a graph automorphism; so the automorphism group of the graph contains a normal subgroup which is the direct product of symmetric groups on the twin classes.

Twins in the hierarchy

Proposition

If χ denotes any graph type in the hierarchy, and G is any non-trivial finite group, then the twin relation on $\chi(G)$ is non-trivial.

Proof.
It is easily checked that two vertices which generate the same cyclic subgroup are closed twins in each of the graphs save possibly the non-generating graph (if G is cyclic). So (excluding this case) we are done unless G has exponent 2. In this case, $\chi(G)$ is a star (if χ is the power graph, enhanced power graph, or deep commuting graph) or a complete graph (in the other two cases).
Cyclic groups are easily dealt with. □

Twin reduction

If two vertices are twins, then we may identify them. This process is known as twin reduction, and it can be iterated. Some interesting properties of a graph are preserved by twin reduction. For example, if F is a graph with trivial twin relation, then twin reduction preserves the property "no induced subgraph isomorphic to F".

Theorem

Given a finite graph Γ, apply twin reduction until no pairs of twins remain. The result is (up to isomorphism) independent of the way the twin reduction is carried out.

The resulting graph is called the cokernel of Γ.

Proof.
The proof is by induction on the number of steps.
If the first step in two sequences of twin reduction involve the same or intersecting pairs of twins, then after one step the graphs are isomorphic, and induction gives the result.
If the first step involves disjoint pairs, then consider the graph Δ obtained by applying both of these identifications to Γ. (Note that the two identifications commute.) By induction each of the original sequences gives the same result as a sequence starting with Δ. □

Cographs and twin reduction

Theorem

A finite graph is a cograph if and only if its cokernel is the 1*-vertex graph.*

Proof.
As we noted, twin reduction cannot create or destroy an induced P_4, so it preserves the property of being a cograph. So we need to show that any cograph with more than one vertex contains a pair of twins.
If Γ is null, this is clear. If Γ is disconnected but not null, then by induction there is a pair of twins in a non-trivial connected component. If Γ is connected, then its complement is disconnected, and so contains a pair of twins; but the property of being twins is preserved by complementation. □

Finite simple groups

Here is the earlier table with the numbers of vertices in the cokernel.

G	$\|G\|$	Pow	EPow	DCom	Com	NGen
A_5	60	1	1	1	1	32
$\mathrm{PSL}(2,7)$	168	1	1	1	44	79
A_6	360	1	1	1	92	167
$\mathrm{PSL}(2,8)$	504	1	1	1	1	128
$\mathrm{PSL}(2,11)$	660	1	1	1	112	244
$\mathrm{PSL}(2,13)$	1092	1	1	1	184	366
$\mathrm{PSL}(2,17)$	2448	1	1	1	308	750
A_7	2520	352	352	352	352	842
$\mathrm{PSL}(2,19)$	3420	1	1	1	344	914
$\mathrm{PSL}(2,16)$	4080	1	1	1	1	784
$\mathrm{PSL}(3,3)$	5616	756	756	808	808	1562
$\mathrm{PSU}(3,3)$	6048	786	534	499	499	1346
$\mathrm{PSL}(2,23)$	6072	1267	1	1	508	1313
$\mathrm{PSL}(2,25)$	7800	1627	1	1	652	1757
M_{11}	7920	1212	1212	1212	1212	2444

A last note on cographs

We have seen hints that cographs and twin reduction are relevant to the study of automorphism groups of the graphs in the hierarchy. So we will revisit this material in the context of automorphism groups later.

Question

Given a graph type X in the hierarchy, for which finite groups G is $\mathrm{X}(G)$ a cograph?

Theorem

The power graph of G is a cograph if and only if there do not exist $g, h \in G$ such that g has order pr and h has order pq (where p, q, r are primes and $p \neq q$) such that

- $g^r = h^q$;
- *if $p = r$, then $g^p \notin \langle h^p \rangle$.*

Split graphs and threshold graphs

The graph Γ is a split graph if its vertex set can be partitioned into two subsets A and B such that A induces a complete graph and B a null graph, with arbitary edges between A and B.
The graph Γ is a threshold graph if its vertices v can be given weights $a(v)$ and there is a threshold t such that v and w are joined if and only if $a(v) + a(w) > t$. Equivalently, a threshold graph is one whose vertices can be enumerated as $(v_1, v_2, \ldots, v_n)$ in such a way that v_i is joined to all or none of its predecessors.

Theorem

- *A graph is a split graph if and only if it contains no induced subgraph isomorphic to C_4, C_5, or $2K_2$.*
- *A graph is a threshold graph if and only if it contains no induced subgraph isomorphic to P_4, C_4, or $2K_2$.*

Here $2K_2$ is the graph with four vertices and two disjoint edges.

Theorem

For a finite group, the following conditions are equivalent:

- $\mathrm{Pow}(G)$ *is a split graph;*
- $\mathrm{Pow}(G)$ *is a threshold graph;*
- $\mathrm{Pow}(G)$ *has no induced subgraph isomorphic to* $2K_2$*;*
- *G does not have subgroups H_1 and H_2 such that each of $H_1 \setminus H_2$ and $H_2 \setminus H_1$ contains an element of order greater than 2;*
- *G is cyclic of prime power order, or an elementary abelian or dihedral 2-group, or cyclic of order $2p$, or dihedral of order $2p^n$ or $4p$, where p is an odd prime.*

Note that this theorem is not restricted to nilpotent groups.

Perfect graphs

The clique number of a graph is the size of the largest induced complete subgraph, while the chromatic number is the least number of colours required to colour the vertices so that adjacent vertices get different colours. The chromatic number is at least as large as the clique number, since a complete subgraph needs as many colours as vertices for a proper colouring.
A graph Γ is called perfect if every induced subgraph of Γ has clique number equal to chromatic number.
It is known that many types of graph are perfect, including bipartite graphs, line graphs of bipartite graphs, and comparability graphs of partial orders.
I review some of the main facts about this class of graphs.

The P_4-structure of a graph Γ is the hypergraph whose hyperedges are the subsets inducing a subgraph P_4. Thus it is the null hypergraph if and only if Γ is a cograph.
The weak, semi-strong and strong perfect graph theorems state:

Theorem

- *(Lovász) The complement of a perfect graph is perfect.*
- *(Reed) If two graphs have isomorphic P_4-structures and one is perfect, then so is the other.*
- *(Chudnovsky et al.) A graph is perfect if and only if it has no induced subgraph which is a cycle of odd length greater than 3 or the complement of one.*

The semi-strong theorem points up a possible connection with cographs and twin reduction, which has not been explored. Could it be true that graphs with isomorphic P_4-structures have cokernels with the same number of vertices?

Perfect graphs in the hierarchy

We will see later that the power graph of a finite group is perfect.
For the other graph types, they may or may not be perfect; there are few results about this, apart from a theorem of Britnell and Gill about the commuting graph. They assume that the group G has a component, a subnormal quasisimple subgroup, and determine all the possible groups which can arise as components if the commuting graph is perfect.

Question

For each graph type χ in the hierarchy other than the power graph, determine the finite groups G for which $\chi(G)$ is perfect.

Lecture 4
The Gruenberg–Kegel graph

The Gruenberg–Kegel graph

I knew Karl Gruenberg well. He was my colleague at Queen Mary, University of London, from the time I moved there in 1986 until his death in 2007. His main work was in the cohomology and integral representation of groups.
I was less well acquainted with Otto Kegel, but he visited Oxford once a week for a term when I was a student there to lecture on locally finite groups.

The Gruenberg–Kegel graph or GK graph for short (sometimes called the prime graph) of a finite group G was introduced by Gruenberg and Kegel in an unpublished manuscript in 1975. They were concerned with the decomposability of the augmentation ideal of the integral group ring of G.
The vertex set of the graph is the set of prime divisors of the order of G (equivalently, by Cauchy's theorem, the set of orders of elements of prime order in G). It has an edge joining p and q if and only if G contains an element of order pq (equivalently, there are commuting elements of orders p and q).
We will see that this small graph has a big influence on the much larger graphs of our hierarchy on the group G. In this lecture I will trace some of these connections.

The theorem

The main theorem of Gruenberg and Kegel was a structure theorem for groups whose GK graph is disconnected. This was published by Williams (a student of Gruenberg) in 1981.
Recall the definitions of Frobenius group and 2-Frobenius group from earlier:

- G is a Frobenius group if it has a non-trivial proper subgroup H such that $H \cap H^g = \{1\}$ for all $g \notin H$. The set of elements in no conjugate of H, together with the identity, form a normal subgroup of G called the Frobenius kernel.
- G is a 2-Frobenius group if it has a normal series $\{1\} \lhd N \lhd M \lhd G$ such that
  - M is a Frobenius group with Frobenius kernel N;
  - G/N is a Frobenius group with Frobenius kernel M/N.

Theorem
Let G be a finite group whose GK-graph is disconnected. Then one of the following holds:

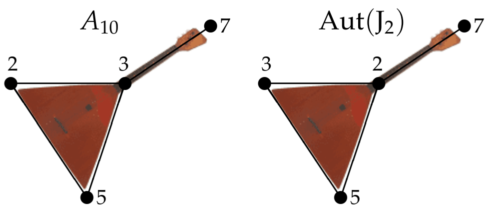
- *G is a Frobenius or 2-Frobenius group;*
- *G is an extension of a nilpotent π-group by a simple group by a π-group, where π is the set of primes in the connected component containing 2.*

Which simple groups can occur in the second conclusion of the theorem? This question was investigated by Williams, though he was unable to deal with groups of Lie type in characteristic 2. The work was completed by Kondrat'ev in 1989, and some errors corrected by Kondrat'ev and Mazurov in 2000.

The GK graph is still a very active area of research. Some of the questions considered are:

- Which groups are characterised by their GK graphs?
- Which groups are characterised by their labelled GK graphs, where the vertices are labelled with the corresponding primes, and how many different labellings can a given graph have?

To mention just one example: the paw, or balalaika, consists of a triangle with a pendant vertex. Among groups whose GK graph is isomorphic to the paw are the alternating group A_{10} and the automorphism group of the sporadic Janko group J_2.



Note that the same primes occur but 2 and 3 swap places.

Graphs in the hierarchy determine the GK graph

Our first result shows that there is a connection between the GK graph and the graphs in our hierarchy.

Theorem

Let χ denote the power graph, enhanced power graph, deep commuting graph, or commuting graph. If G and H are groups with $\chi(G) \cong \chi(H)$, then the Gruenberg–Kegel graphs of G and H are equal.

Proof.

Consider first the enhanced power graph or the commuting graph. A maximal clique in one of these graphs is a maximal cyclic (resp. abelian) subgroup of G. So p and q are joined in the GK graph if and only if there is a maximal clique of the graph having order divisible by pq.

A similar but slightly more elaborate proof works for the deep commuting graph.

Finally, we saw that if the power graphs of G and H are isomorphic, then so are the enhanced power graphs. □

Conversely?

The converse is false, since the GK graph (even with labels) does not determine the order of the group.

However, Natalia Maslova and I recently proved the following theorem:

Theorem

There is a function F on the natural numbers with the property that, if a finite n-vertex graph whose vertices are labelled by pairwise distinct primes is the GK graph of more than $F(n)$ finite groups, then it is the GK graph of infinitely many finite groups.

The function we gave was $O(n^7)$; we believe that better bounds are possible.

It is known that, if there exist infinitely many groups with a given GK graph, then one of these groups has non-trivial soluble radical.

Power graph and enhanced power graph

We previously met the GK graph in this context. Recall that G is an EPPO group if all elements have prime power order. So here is the theorem we saw earlier:

Theorem

Let G be a finite group. Then the following are equivalent:

- $\mathrm{Pow}(G) = \mathrm{EPow}(G)$;
- *G contains no subgroup $C_p \times C_q$ for distinct primes p, q;*
- *G is an EPPO group;*
- *the GK graph of G has no edges.*

I discussed the classification of EPPO groups in Lecture 2. The theorem of Gruenberg and Kegel is an essential ingredient in the proof.

Connectedness

The questions of connectedness of the graphs in the hierarchy will be discussed in much more detail in the next lecture. Here I simply want to point to a couple of connections with the GK graph.

In $\mathrm{Com}(G)$, elements of $Z(G)$ are joined to all other vertices. So it is natural to remove them and ask if what is left is still connected. I will be concerned here with groups satisfying $Z(G) = \{1\}$, so that the only vertex joined to all others in the commuting graph is the identity. Clearly the same is true for graphs below the commuting graph in the hierarchy.

So, for groups G with $Z(G) = \{1\}$, the notations $\mathrm{Com}^-(G)$ and $\mathrm{Pow}^-(G)$ will denote the induced subgraphs of $\mathrm{Com}(G)$ and $\mathrm{Pow}(G)$ on $G \setminus \{1\}$, and call them the reduced commuting and power graphs. (These notations will be generalised in the next lecture.)

The reduced commuting graph

The next theorem was perhaps folklore until it was made explicit in a paper by Morgan and Parker, which I will discuss in the next lecture.

Theorem

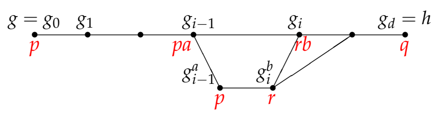
Let G be a finite group with $Z(G) = \{1\}$. Then the reduced commuting graph of G is connected if and only if the GK graph is connected.

The proof does not use the Classification of Finite Simple Groups, or even the structure of groups with disconnected GK graph. I outline it on the next three slides.

Proof

Suppose first that $Z(G) = 1$ and the commuting graph is connected. Let p and q be primes dividing $|G|$. Choose elements g and h of orders p and q respectively, and suppose their distance in the commuting graph is d. We show by induction on d that there is a path from p to q in the GK graph. If $d = 1$, then g and h commute, so gh has order pq, and p is joined to q.

So assume the result for distances less than d, and let $g = g_0, \ldots, g_d = h$ be a path from g to h.
Let i be mimimal such that p does not divide the order of g_i (so $i > 0$). Now some power of g_{i-1}, say g_{i-1}^a, has order p, while a power g_i^b of g_i has prime order $r \neq p$.



Orders written in red under the vertices

The distance from g_i^b to g_d is at most $d - i < d$, so there is a path from r to q in the GK graph. But g_{i-1}^a and g_i^b commute, so p is joined to r.

For the converse, assume that the GK graph is connected. Note first that for every non-identity element g, some power of g has prime order, so it suffices to show that all elements of prime order lie in the same connected component of the commuting graph. Also, since a non-trivial p-group has non-trivial centre, the non-identity elements of any Sylow subgroup lie in a single connected component.
Let C be a connected component. Connectedness of the GK graph shows that C contains a Sylow p-subgroup for every prime p dividing $|G|$. Also, every element of C, acting by conjugation, fixes C. It follows that the normaliser of C is G, and hence that C contains every Sylow subgroup of G, and thus contains all elements of prime order, as required. $\square$

The reduced power graph

A similar result holds in one direction for the reduced power graph of a group with trivial centre:

Proposition

Let G be a group with $Z(G) = \{1\}$. If $\mathrm{Pow}^-(G)$ is connected, then the GK-graph of G is connected.

The proof is left as an exercise for the reader.

Is the power graph a cograph?

The GK graph is also relevant to this question. There is a necessary condition, and a sufficient condition, for the power graph of a group to be a cograph, in terms of the GK graph. However, we will see that there is no necessary and sufficient condition.

Theorem

1. *Suppose that all connected components of the GK graph are singletons (that is, G is an EPPO group). Then the power graph of G is a cograph.*
2. *Suppose that G is insoluble, and that the power graph of G is a cograph. Then every connected component of the GK graph of G except possibly the component containing the prime 2 has size at most 2.*

No necessary and sufficient condition

Consider the two simple groups $\mathrm{PSL}(2,11)$ and M_{11}. The order of each has prime divisors 2, 3, 5 and 11, and each contains elements of order 6 but none of other orders pq for distinct primes p, q.
So in each case the GK graph has an edge $\{2,3\}$ and isolated vertices 5 and 11.
However, $\mathrm{Pow}(\mathrm{PSL}(2,11))$ is a cograph, but $\mathrm{Pow}(M_{11})$ is not. We saw this for $\mathrm{PSL}(2,11)$ earlier; we will discuss M_{11} in more detail later.

Proof.
1. Suppose that G is an EPPO group. Then, in $\mathrm{Pow}^{-}(G)$, there are no edges between elements of distinct prime power orders, so it suffices to show that the induced subgraph on the set of elements of p-power order is a cograph. This is proved by the same argument as that showing that the power graph of a group of prime power order is a p-group.

2. A result in the paper of Williams shows that, if π is a connected component of the GK graph of an insoluble group G which does not contain the prime 2, then G has a nilpotent Hall π-subgroup. By my result with Manna and Mehatari, if the power graph of such a subgroup H is a cograph, then H is either of prime power order or cyclic of order pq, where p and q are distinct primes. □

A difference

This is an isolated result to show that it is possible to say something about the differences between graphs in the hierarchy. We let $(\mathrm{Com}-\mathrm{Pow})(G)$ be the graph whose edges are those of $\mathrm{Com}(G)$ which are not edges of $\mathrm{Pow}(G)$.

Theorem
Suppose that the finite group G satisfies the following conditions:
- *The Gruenberg–Kegel graph of G is connected.*
- *If P is any Sylow subgroup of G, then $Z(P)$ is non-cyclic.*

Then the induced subgraph of $(\mathrm{Com}-\mathrm{Pow})(G)$ on $G \setminus \{1\}$ either has an isolated vertex or is connected.

The hypotheses are very much too strong, and the conclusion rather weak; surely it is possible to do better.

Proof
Let $\Gamma(G)$ denote the induced subgraph of $(\mathrm{Com}-\mathrm{Pow})(G)$ on $G \setminus \{1\}$. Note that, if H is a subgroup of G, then the induced subgraph of $\Gamma(G)$ on $H \setminus \{1\}$ is $\Gamma(H)$.
First we show that, if P is a p-group, then $\Gamma(P)$ is connected. Let $Q \leqslant Z(P)$ with $Q \cong C_p \times C_p$. Then the induced subgraph on $Q \setminus \{1\}$ is complete multipartite with $p+1$ blocks of size $p-1$, corresponding to the cyclic subgroups of Q. So it suffices to show that any element z of $P \setminus \{1\}$ has a neighbour in $Q \setminus \{1\}$.
We see that z commutes with Q since $Q \leqslant Z(P)$; and $\langle z \rangle \cap Q$ is cyclic so there is some element of Q not in this set.
Now let C be a connected component of $\Gamma(G)$ containing an element z of prime order p. Since $\Gamma(G)$ is invariant under $\mathrm{Aut}(G)$, in particular it is normalized by all its elements, so $\langle C \rangle \leqslant N_G(C)$. In particular, C contains a Sylow p-subgroup of G (one containing the given element of order p in C).

If C contains an element of prime order r, and $\{r, s\}$ is an edge of the GK graph, then G contains an element g of order rs, then without loss of generality $g^s \in C$, and g^s is joined to g^r in $\Gamma(G)$, so also $g^r \in C$. Now connectedness of the GK graph shows that C contains a Sylow q-subgroup of G for every prime divisor of $|G|$. Hence $|N_G(C)|$ is divisible by every prime power divisor of $|G|$, whence $N_G(C) = G$.
Finally, let g be any non-identity element of G. Choose a maximal cyclic subgroup K containing g. If $C_G(K) = K$, then the generator of K commutes only with its powers, and is isolated in $\Gamma(G)$. If not, then there is an element of prime order in $C_G(K) \setminus K$. (If $h \in C_G(K) \setminus K$, then $\langle g, h \rangle$ is abelian but not cyclic, so contains a subgroup $\langle g \rangle \times C_m$ for some m; choose an element of prime order in the second factor.) This element is joined to g in the commuting graph but not in the power graph; so $g \in C$.
We conclude that $C = G \setminus \{1\}$, and the proof is done. □

Lecture 5
Connectedness

Connectedness

In this lecture we look more systematically at connectedness of the graphs in the hierarchy, their complements, and related graphs such as intersection graphs of subgroups of various types.
As well as questions of connectedness, we will be interested in bounds for the diameter of connected components.
In the commuting graph, as we have seen, vertices in the centre $Z(G)$ are joined to all others; so, to make the question non-trivial, we remove these vertices. Our first task is to look at the other graphs in the hierarchy and determine which vertices are joined to all others.

Dominating vertices

For each graph type χ in the hierarchy, we let $Z_\chi(G)$ denote the set of vertices which are joined to all other vertices of G.

Theorem

- *$Z_{\text{Pow}}(G)$ is equal to G if G is cyclic of prime power order; or the set consisting of the identity and the generators if G is cyclic of non-prime-power order; or $Z(G)$ if G is a generalized quaternion group; or $\{1\}$ otherwise.*
- *$Z_{\text{EPow}}(G)$ is the product of the Sylow p-subgroups of $Z(G)$ for $p \in \pi$, where π is the set of primes p for which the Sylow p-subgroup of G is cyclic or generalized quaternion; in particular, $Z_{\text{EPow}}(G)$ is cyclic.*
- *$Z_{\text{DCom}}(G)$ is the projection into G of $Z(H)$, where H is a Schur cover of G.*
- *$Z_{\text{Com}}(G) = Z(G)$.*

Note that, in all cases except $Z_{\text{Pow}}(G)$ where G is cyclic of non-prime-power order, $Z_\chi(G)$ is a subgroup of G.
By contrast, $Z_{\text{NGen}}(G)$ is more mysterious. It contains the Frattini subgroup of G, and also the centre, but it may not be a subgroup. For example, if $G = C_6 \times C_6$, then $Z_{\text{NGen}}(G)$ is the union of the Sylow 2- and 3-subgroups of G.
We now formally define the reduced graph $\chi^-(G)$ of each type χ in the hierarchy to be the induced subgraph of $\chi(G)$ on $G \setminus Z_\chi(G)$.

The commuting graph

The question was first investigated for the commuting graph. Early results led Iranmanesh and Jafarzadeh to conjecture that there is an absolute upper bound on the diameter of any connected component of the reduced commuting graph. This was refuted by Giudici and Parker, but Morgan and Parker showed that it is true for groups with trivial centre:

Theorem

- *For any given d there is a 2-group whose reduced commuting graph is connected with diameter greater than d.*
- *Suppose that the finite group G has trivial centre. Then every connected component of its reduced commuting graph has diameter at most 10.*

Power graph and enhanced power graph

For the power graph and enhanced power graph, we note that, if the group G is not cyclic or generalized quaternion, then $Z_{\text{Pow}}(G) = Z_{\text{EPow}}(G) = \{1\}$. For such groups, the question has been considered by several authors.
The next result shows that we have only one rather than two problems to consider.

Proposition

Let G be a group with $Z(G) = \{1\}$. Then the reduced power graph of G is connected if and only if the reduced enhanced power graph of G is connected.

Proof.
The forward implication is trivial; for the reverse, if x and y are joined in the enhanced power graph, they are joined by a path of length 2 in the power graph, whose intermediate vertex is not the identity. □

The non-generating graph

From CFSG, we know that every non-abelian finite simple group is 2-generated. Thus, at least for simple groups, the non-generating graph is not complete.
It is further known that, if G is non-abelian simple, then $Z_{\mathrm{NGen}}(G) = \{1\}$ (we'll see a stronger result shortly); so for the reduced graph, only the identity needs to be deleted.
Shen proved that the reduced non-commuting graph is connected. Recently Saul Freedman proved the following theorem.

Theorem

Let G be a non-abelian finite simple group. Then the reduced non-generating graph of G is connected with diameter at most 5.

It is not currently known whether diameter 5 is realised; the best upper bound is either 4 or 5. These results will be in Saul's thesis and are not yet available.

Complements

We now consider the complements of the graphs in the hierarchy. We begin with a few remarks.

- For any graph type X in the hierarchy, $Z_{\mathrm{X}}(G)$ is the set of elements of G which are isolated in $\mathrm{X}(G)^c$.
- Taking complements reverses the order. So moving down the hierarchy adds edges to the complement.
- As a result, if G is a group and X and Y are graph types with X below Y for which $Z_{\mathrm{X}}(G) = Z_{\mathrm{Y}}(G)$, then connectedness of $\mathrm{Y}(G)^c$ implies connectedness of $\mathrm{X}(G)^c$.

In particular, if G is a non-abelian finite simple group, then for any type X in the hierarchy, $(\mathrm{X}(G)^-)^c$ (the complement of the reduced X graph on G) is connected with diameter at most 5.

Here is a result which applies to the commuting graphs of arbitrary groups.

Proposition

Let Γ be a graph whose vertex set is a group G, and suppose that for any vertex $g \in G$, the closed neighbourhood of g is a subgroup of G. Then the complementary graph has just one connected component of size larger than 1; this component has diameter at most 2.

Proof.

The isolated vertices in the complement of Γ are the vertices whose closed neighbourhood in Γ is the whole of G. Let g_1, g_2 be two elements of G which are not isolated in the complement of Γ. Then $H_1 = \{g_1\} \cup N(g_1)$ and $H_2 = \{g_2\} \cup N(g_2)$ are subgroups of G, where $N(g)$ is the open neighbourhood of g. Since a finite group cannot be written as the union of two proper subgroups (a simple consequence of Lagrange's Theorem), there is a vertex h outside these two subgroups, hence joined to g_1 and g_2 in the complement. $\square$

From this result, it is easy to see that the complement of the deep commuting graph of a group G is connected with diameter 2 apart from isolated vertices. (The Proposition applies because the closed neighbourhood of a vertex is its centraliser.)
With a little more effort, it can be shown that the complement of the power graph is connected apart from isolated vertices.
Two questions remain, which are perhaps not too difficult:

Question

What is the best upper bound for the diameter of the non-trivial component of the complement of the power graph (assuming that G is not cyclic of prime power order)?

Question

What about the enhanced power graph?

Intersection graphs

We can apply some of these results to another type of graph obtained from a finite group G.
Let G be a group which is not trivial and not cyclic of prime order. Let $\mathcal{F}$ be a family of non-trivial proper subgroups of G. The intersection graph of $\mathcal{F}$ is the graph whose vertices are the subgroups in $\mathcal{F}$, with H joined to K whenever $H \cap K \neq \{1\}$.
If we just speak of the intersection graph of G, we take $\mathcal{F}$ to consist of all non-trivial proper subgroups of G.

Bipartite graphs

Let B be a bipartite graph: this means there is a bipartition of the vertex set, a partition into two parts X and Y, such that all edges of the graph have one vertex in X and one in Y.
A connected bipartite graph has a unique bipartition: choose a vertex v, and put all vertices at even distance from v into X and all those at odd distance into Y.
A disconnected bipartite graph has more than one bipartition (indeed, it has 2^{k-1} bipartitions, where k is the number of connected components). But we will choose a fixed bipartition, and regard it as part of the structure of the graph.
Note that X and Y induce null subgraphs of B.

Duality

Let B be a bipartite graph with bipartition $\{X, Y\}$. The halved graphs of B are the graphs Γ and Δ with vertex sets X and Y respectively, such that two vertices in one of these sets are joined by an edge in the corresponding halved graph if and only if they lie at distance 2 in B.
We say that a pair Γ, Δ of graphs are dual to each other if there is a bipartite graph with no isolated vertices such that Γ and Δ are isomorphic to its halved graph.

Dual graphs arise, for example, in incidence geometry. If B is an incidence structure consisting of points and lines, with each point on a line and each line containing a point, then we can represent B as a bipartite graph whose vertices are the points and lines, two vertices joined if they are a point and a line and are incident.
The halved graphs are called the point graph and line graph of the incidence structure.
Duality also arises in the theory of experimental design in statistics, but I will not detour to discuss this.

Connectedness and diameter

Theorem
Let Γ and Δ be dual graphs. Then Γ is connected if and only if Δ is connected. More generally, there is a bijection between the connected components of Γ and those of Δ, with the property that the diameters of corresponding components differ by at most 1.

Proof.
The correspondence is given by the rule that a component of Γ and one of Δ correspond if some vertex of Γ is joined to a vertex of Δ in the graph B.
If two vertices v, w of Γ have distance d in Γ, then they have distance $2d$ in B, and neighbours of v and w have distance $2d-2$, $2d$ or $2d+2$ in B, hence distance $d-1$, d or $d+1$ in Δ. $\square$

A problem

Question
Are there other graph-theoretic properties which can be transferred from a graph to its dual?

Under some (rather strong) regularity conditions, the spectrum of each graph is determined by the spectrum of the other.

First application

Theorem
For any finite group G which is not cyclic, the non-generating graph of G on $G^{\#} = G \setminus \{1\}$ and the intersection graph of G are duals.

Proof.
We define B by the rule that the element $g \in G$ and the non-trivial proper subgroup $H \leq G$ are joined if $g \in H$. Since G is not cyclic, for every $g \neq 1$, the subgroup $\langle g \rangle$ is non-trivial and proper; and any non-trivial subgroup H contains a non-identity element.
Now g and h are joined in $\mathrm{NGen}(G)$ if and only if $\langle g, h \rangle \neq G$; and H and K are joined if and only if there is a non-identity element $g \in H \cap K$. $\square$

The intersection graph

The intersection graph of a finite group was first investigated by Csákány and Pollák, who considered non-simple groups; they determined the groups for which the intersection graph is connected and showed that, in these cases, its diameter is at most 4.
For simple groups, Shen showed that the graph is connected and asked for an upper bound; Herzog *et al.* gave a bound of 64, which was improved to 28 by Ma, and to the best possible 5 by Freedman, who showed that the upper bound is attained only by the Baby Monster and some unitary groups (it is not currently known exactly which).
(Recall here that the diameter of the non-generating graph of a finite simple group is known to be at most 5; no examples with diameter 5 are known.)

A refinement

We don't need to take all subgroups here:

Theorem

If G is non-cyclic, then the induced subgraph of $\mathrm{NGen}(G)$ *on* $G^\#$ *and the intersection graph of maximal proper subgroups of G are dual.*

Proof.

We simply have to note that two elements $g, h \in G$ which don't generate G are contained in some maximal subgroup of G. □

So results about connectedness transfer to this graph as well.

Other examples

Theorem

Suppose that G is non-trivial and $Z(G) = \{1\}$. Then the reduced commuting graph of G and the intersection graph of abelian subgroups (or of maximal abelian subgroups) of G are duals.

Proof.

We simply have to note that two elements which commute are contained in a (maximal) abelian subgroup of G, which is proper since G is nonabelian. □

Recall that, for groups with trivial centre, connectedness of the Gruenberg–Kegel graph is equivalent to connectedness of the reduced commuting graph; so it is also equivalent to connectedness of the intersection graph of (maximal) abelian subgroups.
We also have an upper bound for the diameter of connected components.
A similar result holds for the enhanced power graph and the intersection graph of (maximal) cyclic subgroups. I leave its formulation and proof as an exercise.

Differences

Of course, questions about connectedness of differences between graphs in the hierarchy can also be asked.
The case of $(\mathrm{NGen} - \mathrm{Com})(G)$ has been investigated by Saul Freedman, under the name non-commuting non-generating graph.
Results for nilpotent groups have already appeared:

Theorem

Let G be a finite nilpotent group. Then the induced subgraph of $(\mathrm{NGen} - \mathrm{Com})(G)$ *on* $G \setminus Z(G)$ *is connected of diameter* 2 *or* 3*, apart from isolated vertices. If the diameter is* 3*, then there are no isolated vertices.*

Other differences

We saw a rather weak result for $(\mathrm{Com} - \mathrm{Pow})(G)$ in the last lecture.
But, for the most part, this is unexplored territory.

Lecture 6
Universality

Partial preorders and partial orders

A partial preorder is a binary relation on a set X, which I will denote by $\to$, which is reflexive and transitive; that is,

- for all $x \in X$, $x \to x$ (that is, regarded as a directed graph, there is a loop at each vertex);
- for all $x, y, z \in X$, if $x \to y$ and $y \to z$ then $x \to z$.

A partial preorder is sometimes called a preferential arrangement. If we arrange, say, political candidates in order of preference, there may be some pairs of candidates about whom we are indifferent.

A partial order is a partial preorder which is antisymmetric; that is, it also satisfies the condition

- for all $x, y \in X$, if $x \to y$ and $y \to x$ then $x = y$.

The structure theorem

Given a partial preorder $\to$ on X, define a relation $\equiv$ by the rule that $x \equiv y$ if $x \to y$ and $y \to x$. In the language of preferential arrangements, that means we are indifferent about x and y.

Theorem

Let $\to$ be a partial preorder on X.

- *The relation $\equiv$ is an equivalence relation on X.*
- *If $[x]$ denotes the equivalence class containing x, then the relation on $X/\equiv$ defined by $[x] \preccurlyeq [y]$ if $x \to y$ is independent of the choice of representatives of the equivalence classes, and is a partial order.*

The proof is an exercise.

Comparability graph

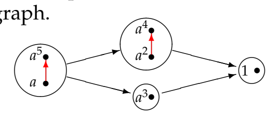
The comparability graph of a partial preorder is the graph on the vertex set X, in which $\{x, y\}$ is an edge if $x \neq y$ and either $x \to y$ or $y \to x$ (or both). (Note that, as usual in graph theory, we have removed loops.)

Theorem

The graph Γ is the comparability graph of a partial order if and only if it is the comparability graph of a partial preorder.

Proof.

The forward implication is clear. For the converse, let $\to$ be a partial preorder, and $\equiv$ the equivalence relation defined on the preceding slide. Refine $\to$ by imposing a total order on each $\equiv$-class. The result is a partial order with the same comparability graph.



□

Dilworth's Theorem

Recall that a graph Γ is perfect if every induced subgraph has clique number equal to chromatic number.

Theorem

(Dilworth) The comparability graph of a partial order, and its complement, are perfect.

Proof

Since a subgraph of a comparability graph is a comparability graph, it suffices to show that a comparability graph (or its complement) has clique number equal to chromatic number. Now a clique in a comparability graph is a chain (a set of pairwise comparable elements), and an independent set is an antichains (a set of pairwise incomprabale elements).

So we have to prove that

- if the size of a maximal chain is c, there is a partition into c antichains;
- if the size of a maximal antichain is a, there is a partition into a chains.

The first is straightforward: partition the points by the length of the longest chain ending at that point. The second is the essence of the theorem. □

By the Weak Perfect Graph Theorem, the second part follows from the first; but of course this postdates Dilworth's Theorem.

The power graph is a comparability graph

Theorem
The power graph of a finite group G is the comparability graph of a partial order.

Proof.
The directed power graph (with loops) is a partial preorder; for

- $x = x^1$;
- if $y = x^m$ and $z = y^n$ then $z = x^{mn}$.

The power graph is its comparability graph. □

Is there anything more special about the class of power graphs?

Universality

In a certain sense, the answer is no:

Theorem
Let the finite graph Γ be the comparability graph of a partial order. Then there is a finite group G such that Γ is isomorphic to an induced subgraph of $\mathrm{Pow}(G)$.

The proof is coming up shortly. But I note here two questions which can be asked about this and other similar results:

Question
Find an upper bound on the function f such that, given any n-vertex comparability graph, there is a group of order at mot $f(n)$ in whose power graph we can embed Γ.

Question
Find an upper bound on the function F such that there exists a group of order at most $F(n)$ whose power graph embeds every comparability graph on at most n vertices.

Proof.
Let Γ be the comparability graph of a partial order $\preccurlyeq$ on $\{1, \ldots, n\}$. Let $p_1, \ldots, p_n$ be distinct primes, and take $G = C_{p_1} \times \cdots \times C_{p_n}$, where $C_{p_i} = \langle a_i \rangle$.
Now let $A_i = \{j \in \{1, \ldots, n\} : j \preccurlyeq i\}$. It is routine to check that $A_j \subseteq A_i$ if and only if $j \preccurlyeq i$, and $A_j = A_i$ if and only if $j = i$.
Now put $b_i = \prod\{a_j : j \in A_i\}$. Check that the following are equivalent:

- b_i and b_j are joined in $\mathrm{Pow}(G)$;
- one of A_i and A_j contains the other;
- i and j are joined in Γ.

□

Other graphs

Theorem
Let X denote one of EPow, DCom, Com, NGen. *Then, given any finite graph Γ, there is a group G such that Γ is isomorphic to an induced subgraph of $\mathrm{X}(G)$.*

In other words, all the other graph types in the hierarchy are universal. So any graph property defined by forbidden induced subgraphs (such as being perfect, a cograph, a threshold graph, etc.) will fail to hold in the graph defined on some groups.
I have previously mentioned some instances of the general problem:

Question
Given a graph type X, and a class $\mathcal{C}$ of graphs defined by forbidden induced subgraphs, determine the groups G for which $\mathrm{X}(G) \in \mathcal{C}$.

Three more

The next three graphs in the hierarchy can be dealt with together, by an argument which also suggests several further open problems.

Theorem
Let Γ be a finite complete graph, whose edges are coloured red, green and blue in any manner. Then there is an embedding of Γ into a finite group G so that

1. *vertices joined by red edges are adjacent in the enhanced power graph;*
2. *vertices joined by green edges are adjacent in the commuting graph but not in the enhanced power graph;*
3. *vertices joined by blue edges are non-adjacent in the commuting graph.*

We can get several results by specialising this construction:

- If we ignore the green/blue distinction, we get an embedding of an arbitrary graph in the enhanced power graph of a group.
- If we ignore the red/green distinction, we get an embedding of an arbitrary graph in the commuting graph of a group.
- If we simply have no green edges, then we have simultaneously embedded the red graph in the enhanced power graph, the deep commuting graph, and the commuting graph.
- If we ignore the red/blue distinction, we get an embedding of an arbitrary graph in the graph $(\mathrm{Com} - \mathrm{EPow})(G)$ for some group G.

Proof

We begin with two observations. First, the direct product of cyclic (resp. abelian) groups of coprime orders is cyclic (resp. abelian).

Second, consider the non-abelian group of order p^3 and exponent p^2, where p is an odd prime:

$$P = \langle a, b \mid a^{p^2} = b^p = 1, [a, b] = a^p \rangle.$$

Any two elements of $\langle a \rangle$ generate a cyclic group; and the group generated by b and x is cyclic if $x = 1$, abelian but not cyclic if $x = a^p$, and non-abelian if $x = a$.

The proof is by induction on the number n of vertices. The result is clearly true if $n = 1$. So let $\{v_1, \ldots, v_n\}$ be the vertex set of Γ, and suppose that we have an embedding of $\{v_1, \ldots, v_{n-1}\}$ into a group G satisfying conditions 1–3 of the theorem.

Choose an odd prime p not dividing $|G|$, and consider the group $P \times G$, where P is as above. Modify the embedding of the first $n - 1$ vertices by replacing v_i by $(1, v_i)$ if $\{v_i, v_n\}$ is red, by (a^p, v_i) if $\{v_i, v_n\}$ is green, and by (a, v_i) if $\{v_i, v_n\}$ is blue. It is easily checked that we still have an embedding of $\{v_1, \ldots, v_{n-1}\}$ satisfying 1–3.

If we now embed v_n as $(b, 1)$, we find that the conditions hold for the remaining pairs as well. □

The non-generating graph

Finally we show the same universality property for the non-generating graph. First, we need a preliminary result.

Theorem

Every graph without isolated vertices and edges can be represented as the intersection graph of a family of sets (that is, the vertices are identified with the sets, two vertices adjacent if the corresponding sets have non-empty intersection).

Proof.

Let E be the edge set of Γ, and for each vertex v, let S_v be the set of edges incident with v. Then

$$S_v \cap S_w = \begin{cases} \{e\} & \text{if } e = \{v, w\}; \\ \emptyset & \text{if } v \text{ and } w \text{ are not joined.} \end{cases}$$

□

Given a graph Γ, we want to embed Γ as an induced subgraph of the non-generating graph of a group. We do this in four steps.

Step 1 Replace Γ by its complement, and represent this graph as an intersection graph.

Step 2 Add some dummy points, each lying in just one of the sets, so that they all have the same cardinality k, with $k \geq 3$. Now add some dummy points in none of the sets so that the cardinality n of the set Ω of points satisfies the conditions that $n > 2k$ and $n - k$ is prime.

Step 3 Now replace each set by its complement. The complements of two subsets of Ω have union Ω if and only if the two sets are disjoint. Thus, each original vertex is now represented by an $(n - k)$-set where two such sets have union Ω if and only if the corresponding vertices are adjacent in Γ.

Step 4 Replace each set by a cyclic permutation on that set, fixing the remaining points. Each of these cycles has odd prime length, so each is an even permutation, and so lies in the alternating group A_n. Let g_v be the permutation corresponding to the vertex v of Γ.

- If v and w are nonadjacent, then the supports of g_v and g_w have union strictly smaller than Ω, so $\langle g_v, g_w \rangle \neq A_n$.
- Suppose v and w are adjacent. Then the supports of g_v and g_w have union Ω, so $H = \langle g_v, g_w \rangle$ is transitive on Ω. Using Jordan's theorem, we conclude that H contains the alternating group A_n. Since it is generated by even permutations, $H = A_n$.

Further properties and parameters

There are a vast number of graph properties and parameters, many of which have been studied for individual graph types in the hierarchy.

For example, a survey of power graphs in 2013 included nearly 100 references, while a survey of developments since then has another nearly 100 references.

I will not even attempt to summarise all this work. Instead, I will say a small amount about cliques (complete subgraphs) and independent sets (null induced subgraphs). The clique number $\omega(\Gamma)$ is the number of vertices in the largest clique, and the independence number $\alpha(\Gamma)$ is the number of vertices in the largest independent set.

Since power graphs are perfect, the clique number of $\mathrm{Pow}(G)$ is equal to its chromatic number, and the independence number is equal to the clique cover number.

For three of our types, the clique number has a group-theoretic interpretation:

Proposition

- *The clique number of* $\mathrm{EPow}(G)$ *is equal to the maximal order of a cyclic subgroup of* G.
- *The clique number of* $\mathrm{Com}(G)$ *is equal to the maximum order of an abelian subgroup of* G.
- *The clique number of* $\mathrm{DCom}(G)$ *is equal to the maximum order of a subgroup of* G *which lifts to an abelian subgroup in a Schur cover of* G.

The only thing that needs comment is that, if a set of elements in a group has the property that any two generate a cyclic group, then the whole set is contained in a cyclic group. The proof is an exercise.

Clique number of the power graph

We begin with cyclic groups. The clique number of $\mathrm{Pow}(C_n)$ was determined by Alireza *et al.* in 2015. I give a slightly different account.

Define a function f on the natural numbers recursively by the rule

- $f(1) = 1$;
- for $n > 1$, $f(n) = \phi(n) + f(n/p)$, where ϕ is Euler's totient function and p is the smallest prime divisor of n.

Theorem

The clique number of $\mathrm{Pow}(C_n)$ *is equal to* $f(n)$.

Proof.

The group C_n has $\phi(n)$ generators; they are dominating vertices in the power graph, so are contained in every maximal clique. It can be shown that the remainder of any maximal clique is contained in a proper subgroup, and the best we can do is to take a maximum-size clique in the largest proper subgroup, the cyclic group of order n/p. Now induction gets us home. $\square$

The function f has a curious property:

Proposition

$f(n) \leq 3\phi(n)$.

In fact, the limit superior of the ratio $f(n)/\phi(n)$ is about $2.6481017597\ldots$. Sean Eberhard has observed that it is equal to

$$\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{1}{p_i - 1},$$

where $p_1, p_2, \ldots$ are the primes in order.

General groups

Rather confusingly, group theorists use the symbol $\omega(G)$ for the *spectrum* of G, the set of orders of elements of G.

Theorem

Let G be a finite group.

- $\omega(\mathrm{EPow}(G)) = \max \omega(G)$.
- $\omega(\mathrm{Pow}(G)) = \max\{f(m) : m \in \omega(G)\}$.

We have seen the first statement; the second holds since a clique in $\mathrm{Pow}(G)$ is a clique in $\mathrm{EPow}(G)$, which is contained in a cyclic subgroup.

The function f is not monotonic, so it is not true that $\omega(\mathrm{Pow}(G)) = f(\omega(\mathrm{EPow}(G)))$. Let $G = \mathrm{PGL}(2, 11)$. The maximal elements of $\omega(G)$ are 10, 11 and 12; so $\omega(\mathrm{EPow}(G)) = 12$. We have $f(10) = f(12) = 9$ and $f(11) = 11$, so $\omega(\mathrm{Pow}(G)) = 11$.

Lecture 7
Onward and upward

I begin this lecture with some comments about extending the hierarchy upwards.
Let $\mathcal{C}$ be a class of groups, which we suppose to be closed under taking subgroups. Then we can define a graph type on a group G by the rule that x and y are joined if and only if $\langle x, y \rangle \in \mathcal{C}$.
We have already seen two examples. Taking $\mathcal{C}$ to be the class of cyclic groups gives the enhanced power graph, while taking $\mathcal{C}$ to be the class of abelian groups gives the commuting graph.
The obvious classes to take are the classes of nilpotent groups and soluble groups, giving the graphs $\mathrm{Nilp}(G)$ and $\mathrm{Sol}(G)$, lying above $\mathrm{Com}(G)$ in the hierarchy. If G is insoluble, then they both lie below $\mathrm{NGen}(G)$.

Minimal excluded groups

For our study, we need the analogue of the Miller–Moreno theorem:

Theorem

- *A minimal non-nilpotent group is 2-generated.*
- *A minimal insoluble group is 2-generated.*

Minimal non-nilpotent groups were classified by Schmidt; these groups are called Schmidt groups. By inspection, they are 2-generated and soluble.
I do not know a complete classification of minimal insoluble groups. But if G is such a group, and S is its soluble radical, then G/S is a minimal (non-abelian) simple group; such groups were classified by Thompson (in his N-group paper), and all are 2-generated (without using CFSG). If we take generators of G/S and lift to G, the resulting elements generate G (by minimality, since the subgroup they generate is insoluble).

$\mathrm{Nilp}(G)$ and $\mathrm{Sol}(G)$

Theorem

1. *For any finite group G, we have* $E(\mathrm{Com}(G)) \subseteq E(\mathrm{Nilp}(G)) \subseteq E(\mathrm{Sol}(G))$.
2. $E(\mathrm{Com}(G)) = E(\mathrm{Nilp}(G))$ *if and only if all the Sylow subgroups of G are abelian.*
3. $E(\mathrm{Nilp}(G)) = E(\mathrm{Sol}(G))$ *if and only if G is nilpotent.*
4. $E(\mathrm{Com}(G)) = E(\mathrm{Sol}(G))$ *if and only if G is abelian.*
5. *If G is non-nilpotent, then* $E(\mathrm{Nilp}(G)) \subseteq E(\mathrm{NGen}(G))$*; equality holds if and only if G is a Schmidt group.*
6. *If G is insoluble, then* $E(\mathrm{Sol}(G)) \subseteq E(\mathrm{NGen}(G))$*; equality holds if and only if G is a minimal insoluble group.*

We have observed the first part already, while parts 5 and 6 follow from the fact that these groups are 2-generated.

Proof of 2 Suppose that $E(\mathrm{Com}(G)) = E(\mathrm{Nilp}(G))$. Then two elements from the same Sylow subgroup of G generate a nilpotent group; hence they commute. Conversely, if the Sylow subgroups are abelian, then a nilpotent subgroup is the product of its Sylow subgroups and hence is abelian.

Proof of 3 Suppose that $E(\mathrm{Nilp}(G)) = E(\mathrm{Sol}(G))$. If G is not nilpotent, it contains a minimal non-nilpotent subgroup, a Schmidt group, which is 2-generated and soluble, hence nilpotent, a contradiction. Conversely, if G is nilpotent, then $\mathrm{Nilp}(G)$ is complete.

Proof of 4 If $\mathrm{Com}(G)$ and $\mathrm{Sol}(G)$ coincide, then G is nilpotent with abelian Sylow subgroups, hence is abelian. The converse is clear.

Dominating vertices

Recall that, if X is a graph type, then $Z_{\mathsf{X}}(G)$ is the set of vertices of $\mathsf{X}(G)$ which are joined to all others.
The first part of the following theorem was proved by Abdollahi and Zarrin, the second by Guralnick *et al.*

Theorem

For any finite group G,

- $Z_{\mathrm{Nilp}}(G)$ *is the hypercentre of G (the last term in the ascending central series for G);*
- $Z_{\mathrm{Sol}}(G)$ *is the soluble radical of G.*

Note that both are subgroups of G.
Now we are all set up for an analysis of these graphs along the lines we have seen for the lower terms in the hierarchy. But not much has been done on this, except for universality.

Universality

We can catch three birds in one net here.
Recall that any graph can be represented as an intersection graph of a linear hypergraph (two sets corresponding to adjacent vertices agreeing in one point). Now take the complement of Γ, represent it in this way, add dummy points so that each set has the same prime cardinality $p > 2$, and replace each set by a cycle with the given set as support.
Two disjoint cycles commute, while two intersecting cycles generate the alternating group A_{2p-1}, which is not soluble.
Hence we have shown that Γ can be embedded in $\mathrm{Com}(G)$, $\mathrm{Nilp}(G)$, and $\mathrm{Sol}(G)$, for some group G (the group generated by all the p-cycles, which is a product of alternating groups).

Indeed, for $p > 3$, we can catch another bird.
Assume that the complement of Γ is connected. (This can be achieved by adding an isolated vertex to Γ if necessary.) Then the group generated by the cycles is the alternating group on the union of the supports of the cycles.
Its Schur multiplier has order 2, so the lift of $C_p \times C_p$ to the Schur cover splits, and so disjoint p-cycles are joined in the deep commuting graph. So we can add $\mathrm{DCom}(G)$ to our tally.

The Engel graph

We define, for each positive integer k, and all $x, y \in G$, the element $[x, _k y]$ of G to be the left-normed commutator of x and k copies of y; more formally,

- $[x, _1 y] = [x, y] = x^{-1}y^{-1}xy$,
- for $k > 1$, $[x, _k y] = [[x, _{k-1} y], y]$.

Abdollahi defined x and y to be adjacent if $[x, _k y] \neq 1$ and $[y, _k x] \neq 1$ for all k. To fit with the earlier philosophy I will redefine it to be the complement of this graph. If we do this then we have a similar situation to that arising with the power graph.
We can define the directed Engel graph to have an arc from x to y if $[y, _k x] = 1$ for some k. Then the Engel graph is the graph in which x and y are joined if there is an arc from one to the other. The directed graph may also have a role to play here.

Zorn showed that, if a finite group G satisfies an Engel identity $[x, _k y] = 1$ for all x, y (for some k), then G is nilpotent; so the finite groups for which the directed Engel graph is complete are the same as those for which the nilpotency graph is complete. (For infinite groups, this is not true, though the result has been shown in a number of special cases.)
So there is a close connection between the Engel graph and the nilpotency graph. But they are not equal in general. For example, in the group S_3, there is an arc of the directed Engel graph from each element of order 3 to each element of order 2, but not in the reverse direction.

Question
What can be said about the relation between the Engel and nilpotency graphs? In particular, in which groups are they equal?

The Engel centre

Question
Which elements of the group G are joined to all others in the Engel graph?

I think the answer should be the Fitting subgroup, $F(G)$, the largest normal nilpotent subgroup of G. It is true that in the directed Engel graph, if $x \in F(G)$, then $x \to y$ for all $y \in G$. For $[y, x] \in F(G)$, and so repeated commutation with x results in the identity.
But I cannot at present prove the converse.

More graphs

A wide generalisation has been considered by Lucchini and Nemmi. Let $\mathfrak{F}$ be a saturated formation of groups. (A formation is a class of groups closed under quotients and subdirect products; the formation $\mathfrak{F}$ is saturated if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$, where $\Phi(G)$ is the Frattini subgroup of G. Now the $\mathcal{F}$-graph of G can be defined by joining x and y if $\langle x, y \rangle \in \mathfrak{F}$.
Their results concern the set of vertices joined to all others in the $\mathfrak{F}$-graph of G (that is, the isolated vertices in the complement), and the connectedness of the complement apart from these isolated vertices. However, time precludes my giving details.

Automorphisms

Because these graphs are so closely connected with the groups they live on, you would expect their automorphism groups to reflect this structure.
If you construct the power graph of A_5, and work out the order of its automorphism group, you come up with the answer

$$668594111536199848062615552000000.$$

What is going on??
After removing the identity (which is fixed by all automorphisms), the graph is a disjoint union of cliques corresponding to the cyclic subgroups: 15 isolated points, 10 cliques of size 2 and 6 of size 4. So we have a normal subgroup n fixing all these, with structure $S_2^{10} \times S_4^6$, and the quotient is $S_{15} \times S_{10} \times S_6$; the product of the orders of these groups is the number quoted earlier.

The group M_{11}

Here is a more interesting example, the sporadic Mathieu group M_{11} of order 7920.
If we remove the identity, and then do closed twin reduction, and then open twin reduction, we reach a twin-free graph, the cokernel of the reduced power graph. It has 1210 vertices, and its automorphism group is exactly M_{11}. In fact this graph is bipartite, and the group acts with four orbits, of sizes 165 (twice), 220 and 660. Lurking in there is a very interesting bipartite graph with blocks of sizes 165 and 220, having diameter and girth equal to 10 (and again automorphism group M_{11}.
It should be said that things are not always so interesting. It often happens that the original group “gets lost in the noise”.

General results

To mention a couple of general results that we have seen implicitly:

Theorem
For each graph type χ in the hierarchy, and any non-trivial group G, the group $\mathrm{Aut}(\chi(G))$ has a non-trivial (usually large) normal subgroup which is a direct product of symmetric groups on the twin classes.

Theorem
The automorphism group of a cograph is built from the trivial group by the operations of direct product and wreath product with a symmetric group.

So, if $X(G)$ is a cograph, then G will almost certainly be “lost in the noise”.

A question

Question
For which graph types χ, and for which groups G, is it true that the automorphism group of the cokernel of $\chi(G)$ is equal to the automorphism group of G?

As noted, this is the case for the power graph of M_{11}.

Infinite groups

There are a number of results about graphs in the hierarchy defined on infinite groups. I begin with one of the best known. This theorem was proved by Bernhard Neumann, answering a question of Paul Erdős.

Theorem
Let G be a group, and suppose that $\mathrm{Com}(G)$ contains no infinite independent set. Then there is a finite upper bound on the size of independent sets in $\mathrm{Com}(G)$.

Neumann formulated the result in terms of cliques in the non-commuting graph.
I will sketch part of the proof, since is is a nice mixture of group theory and graph theory.

Proof.
The proof consists of showing that the hypothesis implies that $Z(G)$ has finite index in G. Now two elements in the same coset of $Z(G)$ commute, so an independent set cannot be larger than $|G : Z(G)|$.
The assertion follows by group-theoretic argument from the following claim:

Every conjugacy class in G is finite.

For if not, let g lie in an infinite conjugacy class, and let S be an infinite set such that the elements $s^{-1}gs$ are all distinct for some x. By Ramsey’s Theorem, this set contains an infinite clique U. But if $u, v \in U$, then

$$[xu, xv] = u^{-1}x^{-1}v^{-1}x^{-1}xuxv = (u^{-1}xu)^{-1}(v^{-1}xv) \neq 1,$$

since u and v commute. But then xU is an infinite independent set, a contradiction. □

Other graphs

If G is an infinite group for which $\mathrm{Pow}(G)$ or $\mathrm{EPow}(G)$ has no infinite independent set, then of course $\mathrm{Com}(G)$ has no infinite independent set, and so $Z(G)$ has finite index in G.
However, the analogue of Neumann's Theorem fails.
Consider first the group C_{p^∞}, which can be defined either as the group of p-power roots of unity in $\mathbb{C}$, or as the group of rationals with p-power denominator in $\mathbb{Q}$ modulo $\mathbb{Z}$. This group has the property that its subgroups are finite cyclic groups of p-power order, one for each power of p. So the power graph is complete.
Incidentally, this shows how far the power graph is from determining the group in the infinite case: indeed, we cannot even determine the prime p from the power graph.
The directed power graph does determine the prime, since the set of elements immediately above the identity in the preorder has cardinality $p-1$.

Now consider the group $G = C_{p^\infty} \times C_{p^\infty}$. It is not hard to show that the power graph of G contains no infinite independent set. However, if a_i and b_i denote elements of order p^i in the two factors, then the set

$$\{(a_0, b_n), (a_1, b_{n-1}), \ldots, (a_n, b_0)\}$$

is an independent set of size $n+1$, for any n.
Nevertheless, something can be proved:

Theorem
Let G be an infinite group. Then the following are equivalent:

- $\mathrm{Pow}(G)$ *has no infinite coclique;*
- $Z(G)$ *has finite index in G and is a direct sum of finitely many p-torsion subgroups of finite rank, for primes p.*

So G is locally finite, a result of Shitov.

For the enhanced power graph, Abdollahi and Hassanabadi proved that the analogue of Neumann's Theorem does hold:

Theorem
Let G be an infinite group. Then the following are equivalent:

- $\mathrm{EPow}(G)$ *has no infinite coclique;*
- *there is a finite upper bound for the size of cocliques in* $\mathrm{EPow}(G)$*;*
- $Z_{\mathrm{EPow}}(G)$ *has finite index in G.*

Recall that $Z_{\mathrm{EPow}}(G)$ is a subgroup of G, called the cyclicizer. It is the set of elements $x \in G$ such that, for all $y \in G$, $\langle x, y\rangle$ is cyclic.

Cliques and colourings

The following striking result holds for the power graph of an infinite group:

Theorem
The power graph of an infinite group has clique number and chromatic number at most countable.

Of course there is no such result for the commuting graph, since there are arbitrarily large abelian groups. We have the following result for the case where the numbers are finite.

Theorem
For an infinite group G, the following conditions are equivalent:

- $\mathrm{Pow}(G)$ *has finite clique number;*
- $\mathrm{Pow}(G)$ *has finite chromatic number;*
- $\mathrm{EPow}(G)$ *has finite clique number;*
- $\mathrm{EPow}(G)$ *has finite chromatic number;*
- *G is a torsion group with finite exponent.*

Proof.
The power graph of an infinite cyclic group $\langle g\rangle$ contains an infinite clique $\{g^{2^n} : n \geq 0\}$. So a group satisfying any of the first four conditions is a torsion group. Now the results are proved just as for finite groups. □

Directing the power graph

We saw that the power graph determines the directed power graph up to isomorphism in the case of a finite group. This fails for infinite groups: the groups C_{p^∞}, for primes p, all have power graph which is countable and complete, but their directed power graphs are all different.
But the result does hold for torsion-free groups. Indeed, a theorem of Zahirović shows clearly the important role played by C_{p^∞}:

Theorem
Let G and H be infinite groups with $\mathrm{Pow}(G) \cong \mathrm{Pow}(H)$. Suppose that G has no subgroup $K \cong C_{p^\infty}$ with the property that, for any cyclic subgroup L of G, either $L \leq K$ or $L \cap K = \{1\}$. Then $\mathrm{DPow}(G) \cong \mathrm{DPow}(H)$.

Lecture 8
Other worlds

Magmas

Many of our graphs can be defined on structures much more general than groups. We can't expect such a rich theory, but maybe there is something to be said.
I will start with a magma, a set with a binary operation (without further restriction). (These objects are sometimes called groupoids, but this term is also used in category theory for a category with all morphisms invertible, so I will avoid it here.)
Sometimes I will write the operation as $x \circ y$, and sometimes I will just concatenate, as is usually done for multiplication in a group.
One can define the commuting graph of an arbitrary magma. We cannot expect to define, say, the deep commuting graph. But what about the power graph?

Power-associative magmas

There are several ways to define powers in a magma. For example, we could set

- $x^1 = x$;
- $x^{n+1} = (x^n) \circ x$ for $n \geq 1$.

But different definitions (for example, $x^{n+1} = x \circ (x^n)$) could give different results. We define a magma to be power-associative if the value of x^n is independent of the definition. This can be expressed by the equations

$$(x^m) \circ (x^n) = x^{m+n} \text{ for } n \in \mathbb{N}.$$

Now in any magma we could define the directed power graph by the rule that $a \to b$ if $b = a^n$ for some $n \in \mathbb{N}$, and the power graph by the rule that $x \sim y$ if either $x \to y$ or $y \to x$. But this is not likely to make much sense unless the magma is power-associative.

Theorem
In a power-associative magma M,

- *the directed power graph is a partial preorder;*
- *the power graph is the comparability graph of a partial order;*
- *the power graph is perfect.*

The proof is immediate.
Now one of our early theorems about groups asserted that if two groups have isomorphic power graphs, then they have isomorphic directed power graphs.

Question
What assumptions on a magma are required for this theorem to hold?

Quasigroups and loops

The Cayley table of a magma M of order n is the $n \times n$ array with rows and columns indexed by M, having (x, y) entry $x \circ y$. (Note that some people reserve the term "Cayley table" for groups, and would call this an "operation table".)
A magma is a quasigroup if it satisfies the left and right division laws; that is, for any a and b, each of the equations $x \circ a = b$ and $a \circ y = b$ has a unique solution. A quasigroup is a loop if it has an identity element.
In terms of the Cayley table, a quasigroup is a magma for which each element occurs once in each row and once in each column of the Cayley table (in other words, the Cayley table is a Latin square). If the quasigroup is a loop, and we order it so that the identity is the first element, then the first row agrees with the row of column labels, and the first column agrees with the column of row labels.

Moufang loops

A group is a loop which satisfies the associative law.
There are two important classes of loops which satisfy a relaxation of the associative law, and so are more general than groups: Moufang loops and Bol loops. I will treat Moufang loops here.
A Moufang loop is a loop satisfying the identity

$$z(x(zy)) = ((zx)z)y$$

(or any one of three equivalent identities).

Theorem
In a Moufang loop M,

- *if $x(yz) = (xy)z$, then the subloop generated by x, y, z satisfies the associative law;*
- *$x(xy) = (xx)y$, $y(xx) = (yx)x$, and $(xy)x = x(yx)$;*
- *any 2-generated subloop is associative.*

Arguably, Moufang loops are so close to groups that they may be expected to have some of the properties of groups.
After proving that the power graph of a group determines the directed power graph up to isomorphism, I conjectured that the same holds for Moufang loops.
This has very recently been proved by Nick Britten:

Theorem

If M and N are Mougang loops whose power graphs are isomorphic, then their directed power graphs are isomorphic.

Here is a sketch of the proof. It follows the corresponding proof for groups. If only the identity is a dominating vertex, the same argument works.

In the group case, if the power graph has other dominating vertices, the group must be cyclic or generalised quaternion. A generalised quaternion group can be characterised as a group in which every commutative subgroup is cyclic.
A generalised octonion loop is a non-associative Moufang loop in which every commutative and associative subloop is cyclic.
A construction due to Chein produces Moufang loops.

Theorem

The following are equivalent for a Moufang loop M:

- *M is a generalized octonion loop;*
- *M is a specific subloop of the unit octonions;*
- *M is a finite Moufang loop of 2-power exponent with a unique element of order 2;*
- *M is produced by Chein's construction.*

Now let M be a Moufang loop whose power graph has a dominating vertex.
The proof shows that, if the order of M is not a 2-power, then it must be a cyclic group; if it is a 2-power, then M is cyclic, generalised quaternion, or generalised octonion, depending on whether the largest complete subgraph has size $|M|$, $|M|/2$ or $|M|/4$.

Question

Can anything similar be done for other classes of loops?

There is a body of research on commuting graphs and power graphs of loops, especially Moufang and Bol loops, which I cannot describe here.
As far as I am aware, the enhanced power graph of a power-associative loop has not been studied, although the definition presents no problems.
It is probably true that the deep commuting graph makes little sense outside the context of groups.

Semigroups and monoids

The other type of magmas of wide interest is semigroups. A semigroup is a magma satisfying the associative law $x(yz) = (xy)z$. A semigroup with an identity element is a monoid.
There are various classes of semigroups which resemble groups to a greater or lesser degree. Perhaps the class which is closest to groups, and so most likely to give an interesting theory, consists of inverse semigroups.
An inverse semigroup is a semigroup in which, for each element x, there is a unique generalised inverse y satisfying $xyx = x$ and $yxy = y$. The element y is denoted by x^*.

Random walks

We saw in the first lecture that the random walk on the commuting graph of a group (with a loop at every vertex) has limiting distribution which is uniform on conjugacy classes.

Question

Are there any classes of magmas beyond groups for which the limiting distribution of the random walk on the commuting graph can be described in terms of the structure of the magma?

Our proof for groups involved the action of the group on itself by conjugation. This can be extended to inverse semigroups, where conjugation by a is the map $x \mapsto a^*xa$, where a^* is the quasi-inverse of a.
So inverse semigroups might be candidates for the above question . . .

Rings

Once we move on to structures with two binary operations, there are more opportunities for defining graphs to reflect the structure.
Recall that a ring has two operations, addition and multiplication, written in the usual way: the ring forms an abelian group with the operation $+$ (the identity and inverse of x are denoted 0 and $-x$), while multiplication is associative and distributive over addition. Important classes of rings are commutative rings and rings with identity (these refer to the multiplication).
In what follows, "ring" will mean "commutative ring with identity".
I will talk about the zero-divisor graph, though other graphs such as the unit graph have been considered.

Finite rings

An ideal in a ring R is the kernel of a ring homomorphism. Thus it is a non-empty subset I closed under addition, with the property that for any $a \in I$ and $r \in R$, we have $ar \in I$.
A ring is local if it has a unique maximal ideal, and semi-local if it has only finitely many. Clearly a finite ring is semi-local.

Theorem
A finite ring is isomorphic to a direct sum of local rings.

This uses some standard results from ring theory. The radical I of a finite ring R is nilpotent, and hence R is complete in the I-adic topology.

The zero-divisor graph

An element a of a ring R is a zero-divisor if $a \neq 0$ and there exists $b \in R$ with $b \neq 0$ such that $ab = 0$.
The zero-divisor graph of R has vertex set the set of zero-divisors in R, with a and b joined if $ab = 0$.
For example, in the ring $\mathbb{Z}/(6)$ of integers mod 6, the zero-divisors are $2, 3, 4$, and the zero-divisor graph is a 3-vertex path.
This graph was introduced by Anderson and Livingston in 1999.

Universality

Theorem
Every finite graph is an induced subgraph of the zero-divisor graph of a finite ring.

Proof.
We use Boolean rings: the elements are all subsets of a set X, with symmetric difference for addition and intersection for multiplication. Now $ab = 0$ if and only if a and b are disjoint. So if we represent the given graph as an intersection graph, it is naturally embedded in the zero-divisor graph of a Boolean ring. □

I am grateful to G. Arun Kumar for this proof.

Local rings

In a finite ring, every non-zero element is either a zero-divisor or invertible. (If multiplication by a is not injective, then a is a zero-divisor; if it is surjective, then a is a unit.) So in a local ring, the zero-divisors are the non-zero elements of the maximal ideal.

Question
Are the zero-divisor graphs of local rings universal (in the previous sense)?

The answer is negative in one special case.

An ideal of a ring is principal if it is generated by a single element.

Theorem
If R is a finite local ring whose maximal ideal is principal, then the zero-divisor graph of R is a threshold graph.

Proof.
Let m generate the maximal ideal. Then every element of R has the form um^i, where u is a unit. There is a minimum i such that $m^i = 0$, say $i = k$; then um^i is joined to vm^j if and only if $i + j \geq k$. So the zero-divisor graph is a threshold graph. □

However, not all finite local rings have their maximal ideals principal, and the zero-divisor graph is not always a threshold graph. The question above remains unanswered.

I hope you have enough information now to begin tackling some of the open questions I have mentioned.
Take a look at my paper on this topic in the *International Journal of Group Theory*, which you can download from

`https://ijgt.ui.ac.ir/article_25608.html`

This paper also contains an extensive bibliography.

Please tell me about anything you manage to find!