Graphs defined on groups

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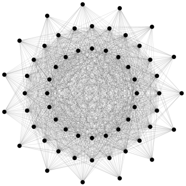

Lecture 1: Some of the players
8 June 2021

Brauer and Fowler

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- Richard Brauer and K. A. Fowler, On groups of even order, *Ann. Math.* **62** (1955), 565–583.

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This immediately suggests the problem of characterising finite simple groups by the centraliser of an involution. This became a major constituent of CFSG.

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This is obviously the distance in the graph with vertex set $G^{\#}$, in which two vertices are joined by an edge if they commute.
This was the first appearance of what is now known as the reduced commuting graph of G.

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Proof.

The subgroup generated by x and y is a dihedral group D_{2n} of order $2n$. Now n must be even, since if it were odd then x and y would generate Sylow subgroups of $\langle x, y \rangle$, and so would be conjugate, contrary to hypothesis. So D_{2n} contains a central involution z, which commutes with both x and y. □

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A simple enough argument, but it shows the blend of group theory and graph theory you should expect in the remainder of this course.

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There are a number of further graphs that have been considered; some of these already have a large literature. These include the power graph, enhanced power graph, deep commuting graph, generating graph, nilpotency graph, solubility graph, and Engel graph.
My interest will be not so much in the individual graphs, as in the relations between them. With a little twist, these graphs form a hierarchy on a given group, with each one contained in the next. This will focus our attention on two things: common properties of the graphs, and how they relate; and properties of further graphs which are formed by differences of the edge sets of two graphs in the hierarchy.

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For more details, see my paper "Graphs defined on groups", to appear in the *International Journal of Group Theory*: the doi is

`10.22108/ijgt.2021.127679.1681`

or (better) get it from

`https://ijgt.ui.ac.ir/article_25608.html`

or you can find a version on the arXiv, `2102.11177`.

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- and, of course, to the London Taught Course Centre for the opportunity to preach about them here.

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A Cayley graph is a graph whose vertex set is a group G and which is invariant under right translations by elements of G. It is not invariant under automorphisms of G except in very special cases. By contrast, the graphs I am discussing are invariant under automorphisms of G, because they are uniquely specified by G, without requiring choosing a generating set.

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Otherwise, notation for groups and graphs will be standard. I will try to explain as I go along, but please ask if you need clarification!

Dramatis Personae, 1: the commuting graph

The commuting graph $\mathrm{Com}(G)$ of G has vertex set G; vertices g and h are joined if and only if $gh = hg$. (This definition would put a loop at every vertex; we silently suppress these.)

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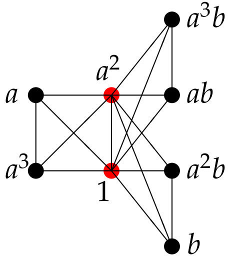
Here are the commuting graphs of the two non-abelian groups of order 8: $D_8 = \langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$.



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- You will recall that Brauer and Fowler removed the identity from the graph; the identity commutes with everything and so is joined to all vertices, thus questions like connectedness (which was important for them) would become trivial. My default is that all graphs are defined on the whole group; when we come to consider connectedness, we first determine which vertices are joined to all others, and then remove them.

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We will denote the commuting graph of G (defined on all of G, but without loops) by $\mathrm{Com}(G)$.

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Theorem (Orbit-counting Lemma)

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Theorem (Orbit-counting Lemma)

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The proof involves constructing a bipartite graph whose vertex set is $G \cup \Omega$, with an edge from $g \in G$ to $x \in \Omega$ if g fixes x. Now counting the number of edges in the graph in two different ways gives the result.

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A small adaptation of the proof of the Orbit-counting Lemma shows that, if we start at a vertex in Ω and take an even number of steps (so that we are back in Ω), the limiting distribution is uniform on orbits – that is, the probability of being at a point $x \in \Omega$ is inversely proportional to the size of the orbit containing x.

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Jerrum called this random walk the Burnside process, since the Orbit-counting Lemma was referred to (incorrectly) by early combinatorial enumerators as "Burnside's Lemma" (it appears without attribution in the second edition of Burnside's book). Peter Neumann traced it back to Cauchy and Frobenius.

Conjugacy classes

A group G acts on itself by conjugation. In this case $\Omega = G$, so we can identify these two sets. Now the group element g fixes x if and only if $gx = xg$. So, for this action, the Burnside process is just a random walk on the commuting graph of G (including the identity, and with a loop at each vertex).

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The importance of this is that some very large groups have very small conjugacy classes. For an extreme example, the symmetric group S_n has order $n!$, but the transpositions form a conjugacy class of size just $n(n-1)/2$. If we are trying to find all conjugacy classes in a large group, the random walk "magnifies" such small classes and makes them more visible.

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Persi Diaconis has used similar ideas to show that the problem of describing conjugacy classes in high-dimensional analogues of Heisenberg groups over finite fields is likely to be hard, since their commuting graphs are arbitrarily complicated.

Dramatis Personae, 2: The power graph and its relatives

The power graph of a group G was first defined by Kelarev and Quinn as a directed graph, with an arc $x \to y$ from x to y whenever y is a power of x. We denote this directed graph by $\mathrm{DPow}(G)$.

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Although pre-empted by Abdollahi, Aalipour *et al.* introduced the enhanced power graph $\mathrm{EPow}(G)$, in which x and y are joined if there exists an element z such that $z \to x$ and $z \to y$ in the directed power graph.

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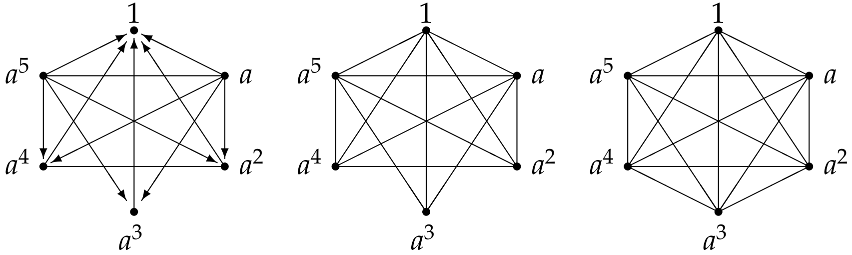
Note that the edge set of the power graph is contained in that of the enhanced power graph (hence the name).

An example: C_6



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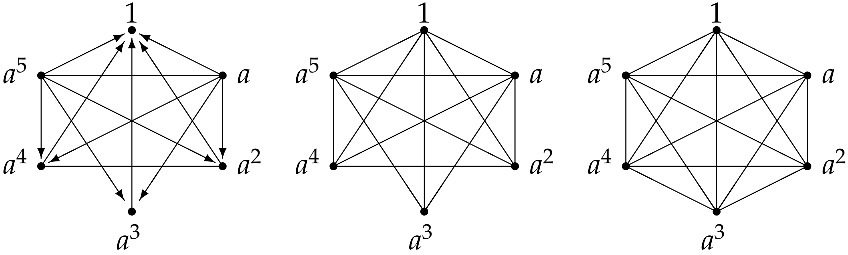

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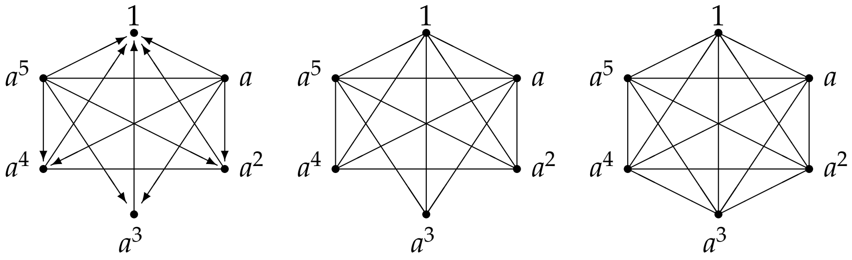

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In the directed power graph, if two elements generate the same cyclic group, then there are arcs in both directions: we represent this by an undirected edge. To get the power graph, we simply ignore the remaining edges.
Note that, in $\mathrm{Pow}(C_6)$, we cannot distinguish between the identity and the two generators; each is joined to all other vertices.

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The implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ come from the definitions. $2 \Rightarrow 1$ was proved by Cameron, and $3 \Rightarrow 1$ by Zahirović. The implication $2 \Rightarrow 1$ does not imply that the directed power graph can be recovered uniquely from the power graph. As we have seen, in the power graph of C_6, the identity and the two generators are indistinguishable, whereas one is a sink and the other two sources in the directed power graph. In $\mathrm{EPow}(C_6)$, all vertices are indistinguishable.

Also, the three equivalent implications do not imply that $G \cong H$: any two groups of exponent 3 with the same order have isomorphic power graphs (consisting of a number of triangles with a common vertex).

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Proof.

Cameron and Ghosh showed that, from the power graph of G, we can reconstruct the numbers of elements of each possible order in G. For abelian groups, this data determines the group up to isomorphism. $\square$

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More on these later.

The generating graph

Instead, we follow a different take on this idea. The generating graph $\mathrm{Gen}(G)$ of G has vertex set G, with vertices x, y joined if $\langle x, y \rangle = G$. Clearly it is a null graph if G cannot be generated by two elements; but we know from CFSG that all finite simple groups can be generated by two elements, so there are interesting examples to consider.

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The generating graph for many interesting groups is fairly dense, as the following result of Burness, Guralnick and Harper shows. We say that a graph has spread k if any k vertices have a common neighbour. Thus, “spread 1” means “no isolated vertices, while “spread 2” means that any two vertices are joined by a path of length 2 (so the diameter is at most 2).

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For reasons which will become clear, I will talk about the non-generating graph $\mathrm{NGen}(G)$, the complement of the generating graph. This also turns out to be connected with small diameter for non-abelian simple G (if the identity is removed).

The deep commuting graph

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Let G be a finite group. A central extension of G is a group H with a normal subgroup Z contained in the centre of H such that $H/Z \cong G$. We regard the epimorphism from H to G as part of the structure of the extension, and call Z the kernel. So we can talk about the inverse images of an element of G in H.

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So it is not obvious that the definition makes sense. But we will see that it is enough to consider one central extension.

Schur covers and Schur multiplier

A central extension H of G with kernel Z is a stem extension of G if $Z \leq Z(H) \cap H'$, where H' is the derived group or commutator subgroup of H.

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The stem extensions of maximal order are called Schur covers of G, and the kernel is the Schur multiplier of G.

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- If we have a presentation of G as F/R, where F is a free group, then the Schur multiplier is $(R \cap F')/[R, F]$.

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As a corollary we see that any two Schur covers of G have isomorphic commuting graphs. This can be proved directly using the notion of isoclinism.

An example

The Klein group $V_4 = C_2 \times C_2$ has two Schur covers, the dihedral and quaternion groups of order 8 (so that its Schur multiplier is C_2). Here is the commuting graph of these groups again:

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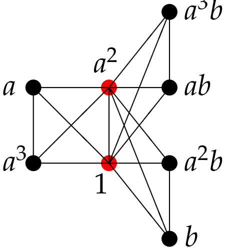

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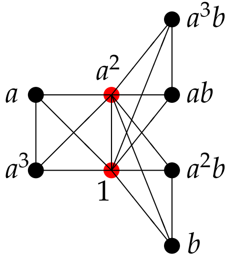

We see that the deep commuting graph of V_4 is the star $K_{1,3}$, even though its commuting graph is the complete graph K_4.

Invariance under automorphisms

For most of the graph types we have defined (the power graph, enhanced power graph, commuting graph, and generating graph), it is clear that any automorphism of the group G induced an automorphism of the corresponding graph on G.

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For most of the graph types we have defined (the power graph, enhanced power graph, commuting graph, and generating graph), it is clear that any automorphism of the group G induced an automorphism of the corresponding graph on G. This is not immediately clear for the deep commuting graph, but it is true in this case. Once we know that our original definition (two vertices joined if their inverse images in every central extension commute) is a good definition, it is clear that the graph is preserved by automorphisms.

Coming up next . . .

In the next lecture, I will consider the graphs we have looked at so far (the power graph, enhanced power graph, deep commuting graph, commuting graph, and non-generating graph), and begin to consider the relations between them.

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As you will see, these graphs form a hierarchy, which can be augmented with the null graph at the bottom and the complete graph at the top.