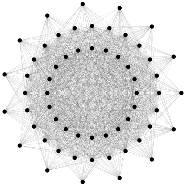
Graphs defined on groups

Peter J. Cameron
University of St Andrews
QMUL (emeritus)

LTCC



Lecture 2: The hierarchy
8 June 2021

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- *Lecture 8: Other worlds*: loops, semigroups, rings

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I will call this the graph hierarchy of G.

Inclusions

Proposition

With the possible exception of $\mathrm{Com}(G)$ *and* $\mathrm{NGen}(G)$*, the edge set of each graph in the hierarchy is contained in that of the next. This holds for* $\mathrm{Com}(G)$ *and* $\mathrm{NGen}(G)$ *if and only if* G *is either non-abelian or not* 2*-generated.*

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Suppose that x and y are joined in the enhanced power graph, so that $\langle x, y\rangle = \langle z\rangle$ for some z. Let H be a central extension of G, with $H/Z \cong G$, and let a, b, c be inverse images of x, y, z in H. Then $\langle Z, a, b\rangle = \langle Z, c\rangle$ is abelian, since Z is central; so a and b commute. □

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For the other gaps, I ignore the deep commuting graph at first, and come back to it later.

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As noted, it is important for us that minimal non-abelian groups are 2-generated.

Frobenius and 2-Frobenius groups

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Frobenius showed that, if N is the set of elements lying in no conjugate of H, together with the identity, then N is a normal subgroup of G, called the Frobenius kernel. Moreover, Thompson showed that the Frobenius kernel is nilpotent, and Zassenhaus determined the structures of Frobenius complements.

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- *every element of G has prime power order.*

Proof.

Commuting elements of distinct prime orders p and q are joined in the enhanced power graph but not in the power graph. Conversely, if x and y are joined in the enhanced power graph but not in the power graph, then $\langle x, y \rangle$ is cyclic but not of prime power order, so it contains an element of order pq for distinct primes p, q. □

Groups with the last property are called EPPO groups.

Classification of EPPO groups

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The non-simple case took longer; following the work of a number of mathematicians, Natalia Maslova and I completed the classification of EPPO groups in a paper now on the arXiv. This is given on the next slide, where $\pi(G)$ denotes the set of prime divisors of $|G|$.

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- $|\pi(G)| = 3$, $G/O_2(G)$ *is* $\mathrm{PSL}_2(2^n)$ *for* $n \in \{2, 3\}$ *and if* $O_2(G) \neq \{1\}$, *then* $O_2(G)$ *is the direct product of minimal normal subgroups of G, each of which is of order* 2^{2n} *and as a* $G/O_2(G)$*-module is isomorphic to the natural* $\mathrm{GF}(2^n)SL(2^n)$*-module.*

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An important ingredient of the classification of EPPO groups is the unpublished theorem of Gruenberg and Kegel, a structure theorem about groups whose Gruenberg–Kegel graph is not connected. More on this later.

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The equivalence of the second and third follows from a theorem of Burnside asserting that groups of p-power order containing no $C_p \times C_p$ subgroup must be cyclic or generalised quaternion. □

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If all Sylow subgroups are cyclic, then G is metacyclic. For let $F(G)$ be the Fitting subgroup of G, the largest normal nilpotent subgroup of G. Then $F(G)$ is a direct product of cyclic groups of coprime orders, so is cyclic. Since it contains its centraliser, $G/F(G)$ embeds into $\mathrm{Aut}(F(G))$, which is abelian with cyclic Sylow subgroups and so is cyclic.

Classification

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If p and q are primes such that $q \mid p-1$, the non-abelian group of order pq (the semidirect product of C_p by C_q) is an example.

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By Glauberman's Z^*-theorem, $\bar{G}$ has a unique central subgroup of order 2, generated by z say. Then $\bar{G}/\langle z \rangle$ has dihedral Sylow 2-subgroups, and so is determined by the Gorenstein–Walter theorem; it must be $\mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$, for an odd prime p.

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There is a unique group $\bar{G}$ for each choice of $\bar{G}/\langle z\rangle$. Indeed, a cohomological argument due to Glauberman shows that any group with dihedral Sylow 2-subgroups has a unique extension containing only one involution (and so having generalised quaternion Sylow 2-subgroups).

The deep commuting graph

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Theorem

Let G be a finite group. Then $\mathrm{DCom}(G) = \mathrm{EPow}(G)$ *if and only if G has the following property: let H be a Schur cover of G, with $H/Z = G$. Then for any subgroup A of G, with B the corresponding subgroup of H (so $Z \leqslant B$ and $B/Z = A$), if B is abelian, then A is cyclic.*

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Proof.

Just a matter of checking the definitions: $\langle x, y \rangle$ is cyclic if x and y are joined in $\mathrm{EPow}(G)$, and their inverse images in a Schur cover generate an abelian group if and only if x and y are joined in $\mathrm{DCom}(G)$. □

An example

Suppose that G is a group and p a prime such that G has a subgroup $H \cong C_p \times C_p$, and p does not divide the order of the Schur multiplier $M(G)$. Then the lift of H to a Schur cover splits over the Schur multiplier, and hence is abelian; but H is not cyclic. So $\mathrm{DCom}(G) \neq \mathrm{EPow}(G)$.

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For example, the Schur multiplier of the alternating group A_n for $n \geq 8$ is cyclic of order 2, and this group contains $C_3 \times C_3$.

The Bogomolov multiplier

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The Schur multiplier of G is denoted by $M(G)$.
Let H be a stem extension of G, and let a, b be the inverse images in H of $x, y \in G$. Clearly, if a and b commute, then so do x and y. If the converse is true, then we say that the extension is commutativity-preserving, or CP for short.

Theorem

Given a finite group G, there is a unique finite abelian group Z such that any CP stem extension of G of largest possible order has kernel Z.

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It has various descriptions. For example, one can define a nonabelian exterior square $G \wedge G$, generated by symbols $x \wedge y$ for $x, y \in G$ subject to the relations

$$(xy) \wedge z = (x^y \wedge z^y)(y \wedge z), \quad x \wedge (yz) = (x \wedge z)(x^z \wedge y^z), \quad x \wedge x = 1.$$

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Then $x \wedge y \mapsto [x, y]$ is a surjective homomorphism from $G \wedge G$ to G' whose kernel is $M(G)$. If we set $M_0(G) = \langle x \wedge y \mid [x, y] = 1 \rangle$; then $B_0(G) \cong M(G)/M_0(G)$.

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Theorem

Let G be a finite group. Then $\mathrm{DCom}(G) = \mathrm{Com}(G)$ *if and only if* $B_0(G) = M(G)$.

Simple groups

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The alternating group A_n for $n \geq 8$ has Schur multiplier of order 2. So for these groups, the enhanced power graph, deep commuting graph, and commuting graph are all unequal.
However, the Schur multiplier of M_{11} is trivial; so the commuting graph and deep commuting graph of this group are equal.

Further examples

It is possible for the Schur and Bogomolov multipliers to be equal when they are both non-trivial. An example is a certain group of order 64 (this is `SmallGroup(64,182)` in the GAP library).

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It has to be admitted that the situation is not perfectly understood …

Differences

Now that we have some kind of description of groups for which two graphs in the hierarchy coincide, a natural question is: if G is a group for which two of these graphs are unequal, what can be said about the graph whose edge set is the difference of the edge sets of the two graphs?

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But there are many more differences that could be explored. Some of these will arise later in this course.

The non-commuting, non-generating graph

One of these difference graphs which has been studied in some detail is the difference between the non-generating graph and the commuting graph (which, we recall, is non-null if and only if G is not a minimal non-abelian group).

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Since this material is not yet available, I will not discuss it here.

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For no two elements of H can generate G.

Coming up next . . .

The next lecture will turn to graph theory. I will discuss cographs (a class of graphs with very nice algorithmic properties) and twin reduction (a simplification process closely connected to cographs).

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For example, the automorphism group of a cograph can be built from the trivial group by the operations "direct product" and "wreath product with a symmetric group".
It turns out that twin reduction is always possible for the groups in the hierarchy, and it is a very interesting question to decide when one of these graphs is actually a cograph.