Graphs defined on groups

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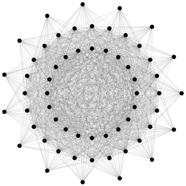

Lecture 3: Cographs and twin reduction
8 June 2021

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A walk from v to w is a sequence $(v_0, v_1, \ldots, v_r)$ of vertices such that $v_0 = v$, $v_r = w$, and v_{i-1} is joined to v_i for $i = 1, \ldots, r$. It is a path if the sequence has no repeated vertices. (If there is a walk from v to w then there is a path.) A graph is connected if there is a path between any two of its vertices. In a connected graph, the distance from v to w is the length (one less than the number of vertices) of the smallest path joining them, and the diameter of the graph is the maximum distance between two vertices.

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The complement of a graph Γ is the graph Γ^c with the same vertex set whose edges are those pairs of vertices which are not edges in Γ.

Subgraphs

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Note that, in our graph hierarchy on a group G, each graph is a spanning subgraph of the next.

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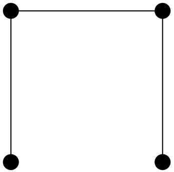
The converse is false, as the graph P_4 (the four-vertex path) shows:



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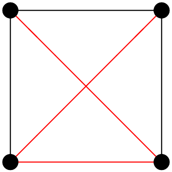

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3. Γ *can be built from the trivial graph by the operations "disjoint union" and "complement".*

A graph satisfying these three conditions is called a cograph.

Proof

$1 \Rightarrow 2$: Suppose that Γ has no induced P_4 but both Γ and its complement are connected, and let Γ be minimal with this property. Then, given any vertex v, if we remove v (that is, take the induced subgraph on $V(\Gamma) \setminus \{v\}$), either the graph or its complement is disconnected, without loss the former.

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I claim that v is joined to all other vertices of Γ. For we can partition $V(\Gamma)$ into two parts A and B so that every path between them passes through Γ. If some vertex u of A were not joined to v, we could take a path of length at least 2 from u to v and an edge from v to a vertex of B, giving an induced path of length 3, contrary to assumption. Similarly for B.

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But now v is an isolated vertex in Γ^c, which is therefore disconnected.

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We see that cographs form the smallest class of graphs containing the 1-vertex graph and closed under complement and disjoint union.

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Order	Groups	Pow	EPow	Com	NGen
1	1	1	1	1	1
2	1	1	1	1	1
4	2	2	2	2	2
8	5	5	5	5	5
16	14	14	14	14	14
32	51	51	51	44	51
64	267	267	267	152	267
128	2328	2328	2328	789	2328

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G	$\lvert G\rvert$	Pow	EPow	DCom	Com	NGen
A_5	60	Y	Y	Y	Y	N
$\mathrm{PSL}(2,7)$	168	Y	Y	Y	N	N
A_6	360	Y	Y	Y	N	N
$\mathrm{PSL}(2,8)$	504	Y	Y	Y	Y	N
$\mathrm{PSL}(2,11)$	660	Y	Y	Y	N	N
$\mathrm{PSL}(2,13)$	1092	Y	Y	Y	N	N
$\mathrm{PSL}(2,17)$	2448	Y	Y	Y	N	N
A_7	2520	N	N	N	N	N
$\mathrm{PSL}(2,19)$	3420	Y	Y	Y	N	N
$\mathrm{PSL}(2,16)$	4080	Y	Y	Y	Y	N
$\mathrm{PSL}(3,3)$	5616	N	N	N	N	N
$\mathrm{PSU}(3,3)$	6048	N	N	N	N	N
$\mathrm{PSL}(2,23)$	6072	N	Y	Y	N	N
$\mathrm{PSL}(2,25)$	7800	N	Y	Y	N	N
M_{11}	7920	N	N	N	N	N

Some explanations

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Proof.

For the first, if G is not 2-generated, then $\mathrm{NGen}(G)$ is complete; if it is 2-generated, then by the Burnside Basis Theorem, any subgroup of index p induces a complete graph, and any two of these complete graphs intersect in the Frattini subgroup $\Phi(G)$ (with index p^2); all other pairs generate.

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For the second, we will see later that the generating graph of a finite simple group and its complement are both connected. $\square$

The power graph of a p-group is a cograph

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Suppose first that (x, y, z) is an induced P_3. In $\mathrm{DPow}(G)$, we cannot have $x \to y \to z$ or $z \to y \to x$, since either would imply $x \sim z$ in $\mathrm{Pow}(G)$. Also we cannot have $y \to x$ and $y \to z$, since then $x, z \in \langle y \rangle$, but the power graph of a cyclic p-group is a complete graph. So we must have $x \to y$ and $z \to y$.

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Now suppose that (x, y, z, w) is a path of length 4, with $x \not\sim z$ and $y \not\sim w$. Then we have $x \to y$ and $z \to y$, and also $y \to z$ and $w \to z$; but these imply $x \to z$, a contradiction. So both induced P_4 and induced C_4 are excluded.

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We will examine the groups $\mathrm{PSL}(2, q)$ on the next slide. Here q is a prime power. If q is a power of 2, let $\{l, m\} = \{q-1, q+1\}$; if q is odd, let $\{l, m\} = \{(q-1)/2, (q+1)/2\}$. Note that $\mathrm{PSL}(2, q)$ has maximal cyclic subgroups of orders l and m.

Proposition

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The odd prime powers up to 500 with the property are 3, 5, 7, 9, 11, 13, 17, 19, 27, 29, 31, 43, 53, 67, 163, 173, 243, 257, 283, 317.

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Question

Are there infinitely many prime powers q *for which the power graph of* $\mathrm{PSL}(2,q)$ *is a cograph?*

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A vertex cannot have both a closed and an open twin. For suppose that u and v are closed twins, and v and w are open twins. Then u and w are not joined (since u's twin v is not joined to w) and also joined (since w's twin v is joined to u), a contradiction.

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A vertex cannot have both a closed and an open twin. For suppose that u and v are closed twins, and v and w are open twins. Then u and w are not joined (since u's twin v is not joined to w) and also joined (since w's twin v is joined to u), a contradiction.
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A vertex cannot have both a closed and an open twin. For suppose that u and v are closed twins, and v and w are open twins. Then u and w are not joined (since u's twin v is not joined to w) and also joined (since w's twin v is joined to u), a contradiction.
Thus, being twins in a graph is an equivalence relation.
Note that interchanging twins (while fixing all other vertices) is a graph automorphism; so the automorphism group of the graph contains a normal subgroup which is the direct product of symmetric groups on the twin classes.

Twins in the hierarchy

Proposition

If χ denotes any graph type in the hierarchy, and G is any non-trivial finite group, then the twin relation on $\chi(G)$ is non-trivial.

Proof.

It is easily checked that two vertices which generate the same cyclic subgroup are closed twins in each of the graphs save possibly the non-generating graph (if G is cyclic). So (excluding this case) we are done unless G has exponent 2. In this case, $\chi(G)$ is a star (if χ is the power graph, enhanced power graph, or deep commuting graph) or a complete graph (in the other two cases).

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Cyclic groups are easily dealt with. □

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Theorem

Given a finite graph Γ, apply twin reduction until no pairs of twins remain. The result is (up to isomorphism) independent of the way the twin reduction is carried out.

The resulting graph is called the cokernel of Γ.

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If the first step involves disjoint pairs, then consider the graph Δ obtained by applying both of these identifications to Γ. (Note that the two identifications commute.) By induction each of the original sequences gives the same result as a sequence starting with Δ. $\square$

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If Γ is null, this is clear. If Γ is disconnected but not null, then by induction there is a pair of twins in a non-trivial connected component. If Γ is connected, then its complement is disconnected, and so contains a pair of twins; but the property of being twins is preserved by complementation. $\square$

Finite simple groups

Here is the earlier table with the numbers of vertices in the cokernel.

Finite simple groups

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G	$\|G\|$	Pow	EPow	DCom	Com	NGen
A_5	60	1	1	1	1	32
$\mathrm{PSL}(2,7)$	168	1	1	1	44	79
A_6	360	1	1	1	92	167
$\mathrm{PSL}(2,8)$	504	1	1	1	1	128
$\mathrm{PSL}(2,11)$	660	1	1	1	112	244
$\mathrm{PSL}(2,13)$	1092	1	1	1	184	366
$\mathrm{PSL}(2,17)$	2448	1	1	1	308	750
A_7	2520	352	352	352	352	842
$\mathrm{PSL}(2,19)$	3420	1	1	1	344	914
$\mathrm{PSL}(2,16)$	4080	1	1	1	1	784
$\mathrm{PSL}(3,3)$	5616	756	756	808	808	1562
$\mathrm{PSU}(3,3)$	6048	786	534	499	499	1346
$\mathrm{PSL}(2,23)$	6072	1267	1	1	508	1313
$\mathrm{PSL}(2,25)$	7800	1627	1	1	652	1757
M_{11}	7920	1212	1212	1212	1212	2444

A last note on cographs

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Question

Given a graph type X in the hierarchy, for which finite groups G is $\mathrm{X}(G)$ a cograph?

Theorem

The power graph of G is a cograph if and only if there do not exist $g, h \in G$ such that g has order pr and h has order pq (where p, q, r are primes and $p \neq q$) such that

- $g^r = h^q$;
- *if* $p = r$, *then* $g^p \notin \langle h^p \rangle$.

Split graphs and threshold graphs

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Here $2K_2$ is the graph with four vertices and two disjoint edges.

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Note that this theorem is not restricted to nilpotent groups.

Perfect graphs

The clique number of a graph is the size of the largest induced complete subgraph, while the chromatic number is the least number of colours required to colour the vertices so that adjacent vertices get different colours. The chromatic number is at least as large as the clique number, since a complete subgraph needs as many colours as vertices for a proper colouring.

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I review some of the main facts about this class of graphs.

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The semi-strong theorem points up a possible connection with cographs and twin reduction, which has not been explored. Could it be true that graphs with isomorphic P_4-structures have cokernels with the same number of vertices?

Perfect graphs in the hierarchy

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Question

For each graph type X in the hierarchy other than the power graph, determine the finite groups G for which $X(G)$ is perfect.

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This small graph has a powerful influence over the much larger graphs in the hierarchy, as we will see.