Graphs defined on groups

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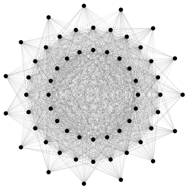

Lecture 4: The Gruenberg–Kegel graph
8 June 2021

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I was less well acquainted with Otto Kegel, but he visited Oxford once a week for a term when I was a student there to lecture on locally finite groups.

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The vertex set of the graph is the set of prime divisors of the order of G (equivalently, by Cauchy's theorem, the set of orders of elements of prime order in G). It has an edge joining p and q if and only if G contains an element of order pq (equivalently, there are commuting elements of orders p and q).

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We will see that this small graph has a big influence on the much larger graphs of our hierarchy on the group G. In this lecture I will trace some of these connections.

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Which simple groups can occur in the second conclusion of the theorem? This question was investigated by Williams, though he was unable to deal with groups of Lie type in characteristic 2. The work was completed by Kondrat'ev in 1989, and some errors corrected by Kondrat'ev and Mazurov in 2000.

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- Which groups are characterised by their labelled GK graphs, where the vertices are labelled with the corresponding primes, and how many different labellings can a given graph have?

To mention just one example: the paw, or balalaika, consists of a triangle with a pendant vertex. Among groups whose GK graph is isomorphic to the paw are the alternating group A_{10} and the automorphism group of the sporadic Janko group J_2.

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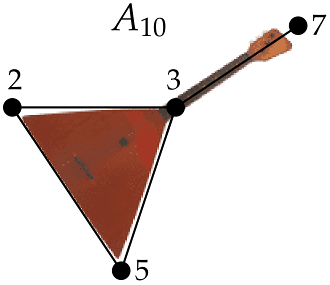

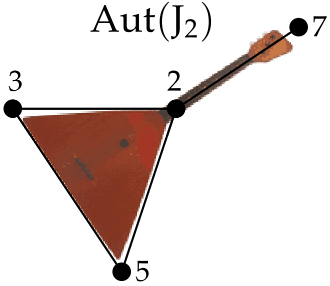

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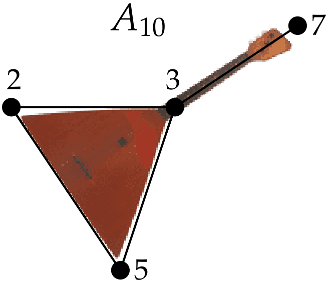

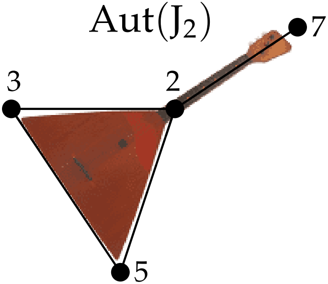

Note that the same primes occur but 2 and 3 swap places.

Graphs in the hierarchy determine the GK graph

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Theorem

Let χ denote the power graph, enhanced power graph, deep commuting graph, or commuting graph. If G and H are groups with $\chi(G) \cong \chi(H)$, then the Gruenberg–Kegel graphs of G and H are equal.

Proof.

Consider first the enhanced power graph or the commuting graph. A maximal clique in one of these graphs is a maximal cyclic (resp. abelian) subgroup of G. So p and q are joined in the GK graph if and only if there is a maximal clique of the graph having order divisible by pq.

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A similar but slightly more elaborate proof works for the deep commuting graph.
Finally, we saw that if the power graphs of G and H are isomorphic, then so are the enhanced power graphs. □

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It is known that, if there exist infinitely many groups with a given GK graph, then one of these groups has non-trivial soluble radical.

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I discussed the classification of EPPO groups in Lecture 2. The theorem of Gruenberg and Kegel is an essential ingredient in the proof.

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So, for groups G with $Z(G) = \{1\}$, the notations $\mathrm{Com}^-(G)$ and $\mathrm{Pow}^-(G)$ will denote the induced subgraphs of $\mathrm{Com}(G)$ and $\mathrm{Pow}(G)$ on $G \setminus \{1\}$, and call them the reduced commuting and power graphs. (These notations will be generalised in the next lecture.)

The reduced commuting graph

The next theorem was perhaps folklore until it was made explicit in a paper by Morgan and Parker, which I will discuss in the next lecture.

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Let G be a finite group with $Z(G) = \{1\}$. Then the reduced commuting graph of G is connected if and only if the GK graph is connected.

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The proof does not use the Classification of Finite Simple Groups, or even the structure of groups with disconnected GK graph. I outline it on the next three slides.

Proof

Suppose first that $Z(G) = 1$ and the commuting graph is connected. Let p and q be primes dividing $|G|$. Choose elements g and h of orders p and q respectively, and suppose their distance in the commuting graph is d. We show by induction on d that there is a path from p to q in the GK graph.

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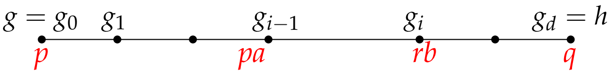

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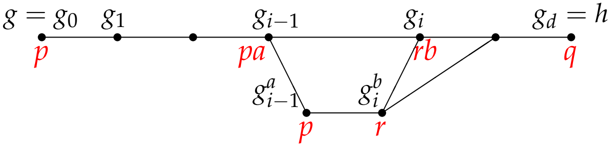

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The distance from g_i^b to g_d is at most $d - i < d$, so there is a path from r to q in the GK graph. But g_{i-1}^a and g_i^b commute, so p is joined to r.

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Let C be a connected component. Connectedness of the GK graph shows that C contains a Sylow p-subgroup for every prime p dividing $|G|$. Also, every element of C, acting by conjugation, fixes C. It follows that the normaliser of C is G, and hence that C contains every Sylow subgroup of G, and thus contains all elements of prime order, as required. □

The reduced power graph

A similar result holds in one direction for the reduced power graph of a group with trivial centre:

Proposition

Let G be a group with $Z(G) = \{1\}$. If $\mathrm{Pow}^-(G)$ is connected, then the GK-graph of G is connected.

The proof is left as an exercise for the reader.

Is the power graph a cograph?

The GK graph is also relevant to this question. There is a necessary condition, and a sufficient condition, for the power graph of a group to be a cograph, in terms of the GK graph. However, we will see that there is no necessary and sufficient condition.

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Theorem

1. *Suppose that all connected components of the GK graph are singletons (that is, G is an EPPO group). Then the power graph of G is a cograph.*
2. *Suppose that G is insoluble, and that the power graph of G is a cograph. Then every connected component of the GK graph of G except possibly the component containing the prime 2 has size at most 2.*

No necessary and sufficient condition

Consider the two simple groups $\mathrm{PSL}(2, 11)$ and M_{11}. The order of each has prime divisors 2, 3, 5 and 11, and each contains elements of order 6 but none of other orders pq for distinct primes p, q.

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So in each case the GK graph has an edge $\{2, 3\}$ and isolated vertices 5 and 11.
However, $\mathrm{Pow}(\mathrm{PSL}(2, 11))$ is a cograph, but $\mathrm{Pow}(M_{11})$ is not.
We saw this for $\mathrm{PSL}(2, 11)$ earlier; we will discuss M_{11} in more detail later.

Proof.
1. Suppose that G is an EPPO group. Then, in $\mathrm{Pow}^{-}(G)$, there are no edges between elements of distinct prime power orders, so it suffices to show that the induced subgraph on the set of elements of p-power order is a cograph. This is proved by the same argument as that showing that the power graph of a group of prime power order is a p-group.

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2. A result in the paper of Williams shows that, if π is a connected component of the GK graph of an insoluble group G which does not contain the prime 2, then G has a nilpotent Hall π-subgroup. By my result with Manna and Mehatari, if the power graph of such a subgroup H is a cograph, then H is either of prime power order or cyclic of order pq, where p and q are distinct primes. $\square$

A difference

This is an isolated result to show that it is possible to say something about the differences between graphs in the hierarchy. We let $(\mathrm{Com} - \mathrm{Pow})(G)$ be the graph whose edges are those of $\mathrm{Com}(G)$ which are not edges of $\mathrm{Pow}(G)$.

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Then the induced subgraph of $(\mathrm{Com} - \mathrm{Pow})(G)$ on $G \setminus \{1\}$ either has an isolated vertex or is connected.

The hypotheses are very much too strong, and the conclusion rather weak; surely it is possible to do better.

Proof

Let $\Gamma(G)$ denote the induced subgraph of $(\mathrm{Com} - \mathrm{Pow})(G)$ on $G \setminus \{1\}$. Note that, if H is a subgroup of G, then the induced subgraph of $\Gamma(G)$ on $H \setminus \{1\}$ is $\Gamma(H)$.

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First we show that, if P is a p-group, then $\Gamma(P)$ is connected. Let $Q \leqslant Z(P)$ with $Q \cong C_p \times C_p$. Then the induced subgraph on $Q \setminus \{1\}$ is complete multipartite with $p+1$ blocks of size $p-1$, corresponding to the cyclic subgroups of Q. So it suffices to show that any element z of $P \setminus \{1\}$ has a neighbour in $Q \setminus \{1\}$. We see that z commutes with Q since $Q \leqslant Z(P)$; and $\langle z \rangle \cap Q$ is cyclic so there is some element of Q not in this set.

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First we show that, if P is a p-group, then $\Gamma(P)$ is connected. Let $Q \leqslant Z(P)$ with $Q \cong C_p \times C_p$. Then the induced subgraph on $Q \setminus \{1\}$ is complete multipartite with $p+1$ blocks of size $p-1$, corresponding to the cyclic subgroups of Q. So it suffices to show that any element z of $P \setminus \{1\}$ has a neighbour in $Q \setminus \{1\}$. We see that z commutes with Q since $Q \leqslant Z(P)$; and $\langle z \rangle \cap Q$ is cyclic so there is some element of Q not in this set.

Now let C be a connected component of $\Gamma(G)$ containing an element z of prime order p. Since $\Gamma(G)$ is invariant under $\mathrm{Aut}(G)$, in particular it is normalized by all its elements, so $\langle C \rangle \leqslant N_G(C)$. In particular, C contains a Sylow p-subgroup of G (one containing the given element of order p in C).

If C contains an element of prime order r, and $\{r, s\}$ is an edge of the GK graph, then G contains an element g of order rs, then without loss of generality $g^s \in C$, and g^s is joined to g^r in $\Gamma(G)$, so also $g^r \in C$. Now connectedness of the GK graph shows that C contains a Sylow q-subgroup of G for every prime divisor of $|G|$. Hence $|N_G(C)|$ is divisible by every prime power divisor of $|G|$, whence $N_G(C) = G$.

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Finally, let g be any non-identity element of G. Choose a maximal cyclic subgroup K containing g. If $C_G(K) = K$, then the generator of K commutes only with its powers, and is isolated in $\Gamma(G)$. If not, then there is an element of prime order in $C_G(K) \setminus K$. (If $h \in C_G(K) \setminus K$, then $\langle g, h \rangle$ is abelian but not cyclic, so contains a subgroup $\langle g \rangle \times C_m$ for some m; choose an element of prime order in the second factor.) This element is joined to g in the commuting graph but not in the power graph; so $g \in C$. We conclude that $C = G \setminus \{1\}$, and the proof is done. □

Coming up next . . .

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In the next section we will examine this more systematically.