Graphs defined on groups

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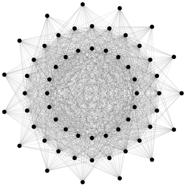

Lecture 5: Connectedness
9 June 2021

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As well as questions of connectedness, we will be interested in bounds for the diameter of connected components.

In the commuting graph, as we have seen, vertices in the centre $Z(G)$ are joined to all others; so, to make the question non-trivial, we remove these vertices. Our first task is to look at the other graphs in the hierarchy and determine which vertices are joined to all others.

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- *$Z_{\mathrm{Pow}}(G)$ is equal to G if G is cyclic of prime power order; or the set consisting of the identity and the generators if G is cyclic of non-prime-power order; or $Z(G)$ if G is a generalized quaternion group; or $\{1\}$ otherwise.*

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- *$Z_{\mathrm{EPow}}(G)$ is the product of the Sylow p-subgroups of $Z(G)$ for $p \in \pi$, where π is the set of primes p for which the Sylow p-subgroup of G is cyclic or generalized quaternion; in particular, $Z_{\mathrm{EPow}}(G)$ is cyclic.*

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- $Z_{\mathrm{DCom}}(G)$ *is the projection into* G *of* $Z(H)$*, where* H *is a Schur cover of* G*.*
- $Z_{\mathrm{Com}}(G) = Z(G)$*.*

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By contrast, $Z_{\mathrm{NGen}}(G)$ is more mysterious. It contains the Frattini subgroup of G, and also the centre, but it may not be a subgroup. For example, if $G = C_6 \times C_6$, then $Z_{\mathrm{NGen}}(G)$ is the union of the Sylow 2- and 3-subgroups of G.

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We now formally define the reduced graph $X^-(G)$ of each type X in the hierarchy to be the induced subgraph of $X(G)$ on $G \setminus Z_X(G)$.

The commuting graph

The question was first investigated for the commuting graph. Early results led Iranmanesh and Jafarzadeh to conjecture that there is an absolute upper bound on the diameter of any connected component of the reduced commuting graph. This was refuted by Giudici and Parker, but Morgan and Parker showed that it is true for groups with trivial centre:

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Theorem

- *For any given d there is a 2-group whose reduced commuting graph is connected with diameter greater than d.*
- *Suppose that the finite group G has trivial centre. Then every connected component of its reduced commuting graph has diameter at most 10.*

Power graph and enhanced power graph

For the power graph and enhanced power graph, we note that, if the group G is not cyclic or generalized quaternion, then $Z_{\mathrm{Pow}}(G) = Z_{\mathrm{EPow}}(G) = \{1\}$. For such groups, the question has been considered by several authors.

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Proof.

The forward implication is trivial; for the reverse, if x and y are joined in the enhanced power graph, they are joined by a path of length 2 in the power graph, whose intermediate vertex is not the identity. □

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Theorem

Let G be a non-abelian finite simple group. Then the reduced non-generating graph of G is connected with diameter at most 5.

It is not currently known whether diameter 5 is realised; the best upper bound is either 4 or 5. These results will be in Saul's thesis and are not yet available.

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- For any graph type X in the hierarchy, $Z_{\mathrm{X}}(G)$ is the set of elements of G which are isolated in $\mathrm{X}(G)^c$.
- Taking complements reverses the order. So moving down the hierarchy adds edges to the complement.
- As a result, if G is a group and X and Y are graph types with X below Y for which $Z_{\mathrm{X}}(G) = Z_{\mathrm{Y}}(G)$, then connectedness of $\mathrm{Y}(G)^c$ implies connectedness of $\mathrm{X}(G)^c$.

In particular, if G is a non-abelian finite simple group, then for any type X in the hierarchy, $(\mathrm{X}(G)^-)^c$ (the complement of the reduced X graph on G) is connected with diameter at most 5.

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Let Γ be a graph whose vertex set is a group G, and suppose that for any vertex $g \in G$, the closed neighbourhood of g is a subgroup of G. Then the complementary graph has just one connected component of size larger than 1; this component has diameter at most 2.

Proof.

The isolated vertices in the complement of Γ are the vertices whose closed neighbourhood in Γ is the whole of G. Let g_1, g_2 be two elements of G which are not isolated in the complement of Γ. Then $H_1 = \{g_1\} \cup N(g_1)$ and $H_2 = \{g_2\} \cup N(g_2)$ are subgroups of G, where $N(g)$ is the open neighbourhood of g. Since a finite group cannot be written as the union of two proper subgroups (a simple consequence of Lagrange's Theorem), there is a vertex h outside these two subgroups, hence joined to g_1 and g_2 in the complement. □

From this result, it is easy to see that the complement of the deep commuting graph of a group G is connected with diameter 2 apart from isolated vertices. (The Proposition applies because the closed neighbourhood of a vertex is its centraliser.)

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What is the best upper bound for the diameter of the non-trivial component of the complement of the power graph (assuming that G is not cyclic of prime power order)?

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What about the enhanced power graph?

Intersection graphs

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The intersection graph of $\mathcal{F}$ is the graph whose vertices are the subgroups in $\mathcal{F}$, with H joined to K whenever $H \cap K \neq \{1\}$.
If we just speak of the intersection graph of G, we take $\mathcal{F}$ to consist of all non-trivial proper subgroups of G.

Bipartite graphs

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Note that X and Y induce null subgraphs of B.

Duality

Let B be a bipartite graph with bipartition $\{X, Y\}$. The halved graphs of B are the graphs Γ and Δ with vertex sets X and Y respectively, such that two vertices in one of these sets are joined by an edge in the corresponding halved graph if and only if they lie at distance 2 in B.

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We say that a pair Γ, Δ of graphs are dual to each other if there is a bipartite graph with no isolated vertices such that Γ and Δ are isomorphic to its halved graph.

Dual graphs arise, for example, in incidence geometry. If B is an incidence structure consisting of points and lines, with each point on a line and each line containing a point, then we can represent B as a bipartite graph whose vertices are the points and lines, two vertices joined if they are a point and a line and are incident.

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Duality also arises in the theory of experimental design in statistics, but I will not detour to discuss this.

Connectedness and diameter

Theorem

Let Γ and Δ be dual graphs. Then Γ is connected if and only if Δ is connected. More generally, there is a bijection between the connected components of Γ and those of Δ, with the property that the diameters of corresponding components differ by at most 1.

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If two vertices v, w of Γ have distance d in Γ, then they have distance $2d$ in B, and neighbours of v and w have distance $2d - 2$, $2d$ or $2d + 2$ in B, hence distance $d - 1$, d or $d + 1$ in Δ. □

A problem

Question

Are there other graph-theoretic properties which can be transferred from a graph to its dual?

Under some (rather strong) regularity conditions, the spectrum of each graph is determined by the spectrum of the other.

First application

Theorem

For any finite group G which is not cyclic, the non-generating graph of G on $G^\# = G \setminus \{1\}$ and the intersection graph of G are duals.

Proof.

We define B by the rule that the element $g \in G$ and the non-trivial proper subgroup $H \leq G$ are joined if $g \in H$. Since G is not cyclic, for every $g \neq 1$, the subgroup $\langle g \rangle$ is non-trivial and proper; and any non-trivial subgroup H contains a non-identity element.

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Now g and h are joined in $\mathrm{NGen}(G)$ if and only if $\langle g, h \rangle \neq G$; and H and K are joined if and only if there is a non-identity element $g \in H \cap K$. □

The intersection graph

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(Recall here that the diameter of the non-generating graph of a finite simple group is known to be at most 5; no examples with diameter 5 are known.)

A refinement

We don't need to take all subgroups here:

Theorem

If G is non-cyclic, then the induced subgraph of $\mathrm{NGen}(G)$ on $G^\#$ and the intersection graph of maximal proper subgroups of G are dual.

Proof.

We simply have to note that two elements $g, h \in G$ which don't generate G are contained in some maximal subgroup of G. □

So results about connectedness transfer to this graph as well.

Other examples

Theorem

Suppose that G is non-trivial and $Z(G) = \{1\}$. *Then the reduced commuting graph of G and the intersection graph of abelian subgroups (or of maximal abelian subgroups) of G are duals.*

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Proof.

We simply have to note that two elements which commute are contained in a (maximal) abelian subgroup of G, which is proper since G is nonabelian. □

Recall that, for groups with trivial centre, connectedness of the Gruenberg–Kegel graph is equivalent to connectedness of the reduced commuting graph; so it is also equivalent to connectedness of the intersection graph of (maximal) abelian subgroups.

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A similar result holds for the enhanced power graph and the intersection graph of (maximal) cyclic subgroups. I leave its formulation and proof as an exercise.

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Results for nilpotent groups have already appeared:

Theorem

Let G be a finite nilpotent group. Then the induced subgraph of $(\mathrm{NGen} - \mathrm{Com})(G)$ on $G \setminus Z(G)$ is connected of diameter 2 or 3, apart from isolated vertices. If the diameter is 3, then there are no isolated vertices.

Other differences

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But, for the most part, this is unexplored territory.

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In brief, the power graph is the comparability graph of a partial order, but every such comparability graph is embeddable in the power graph of some group; the other graph types in the hierarchy are universal, in the sense that all finite graphs are embeddable.

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We will also look briefly at other graph parameters.