Graphs defined on groups

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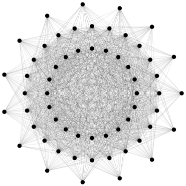

Lecture 6: Universality
9 June 2021

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A partial order is a partial preorder which is antisymmetric; that is, it also satisfies the condition

- for all $x, y \in X$, if $x \to y$ and $y \to x$ then $x = y$.

The structure theorem

Given a partial preorder $\to$ on X, define a relation $\equiv$ by the rule that $x \equiv y$ if $x \to y$ and $y \to x$. In the language of preferential arrangements, that means we are indifferent about x and y.

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Let $\to$ be a partial preorder on X.

- *The relation $\equiv$ is an equivalence relation on X.*
- *If $[x]$ denotes the equivalence class containing x, then the relation on $X/\equiv$ defined by $[x] \preccurlyeq [y]$ if $x \to y$ is independent of the choice of representatives of the equivalence classes, and is a partial order.*

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The proof is an exercise.

Comparability graph

The comparability graph of a partial preorder is the graph on the vertex set X, in which $\{x, y\}$ is an edge if $x \neq y$ and either $x \to y$ or $y \to x$ (or both). (Note that, as usual in graph theory, we have removed loops.)

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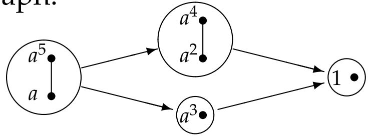
Proof.

The forward implication is clear. For the converse, let $\to$ be a partial preorder, and $\equiv$ the equivalence relation defined on the preceding slide. Refine $\to$ by imposing a total order on each $\equiv$-class. The result is a partial order with the same comparability graph.



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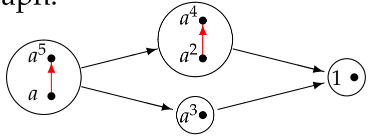

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Now a clique in a comparability graph is a chain (a set of pairwise comparable elements), and an independent set is an antichain (a set of pairwise incomprabale elements).

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By the Weak Perfect Graph Theorem, the second part follows from the first; but of course this postdates Dilworth's Theorem.

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Is there anything more special about the class of power graphs?

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Theorem

Let the finite graph Γ be the comparability graph of a partial order. Then there is a finite group G such that Γ is isomorphic to an induced subgraph of $\mathrm{Pow}(G)$.

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Find an upper bound on the function F such that there exists a group of order at most $F(n)$ whose power graph embeds every comparability graph on at most n vertices.

Proof.
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Now let $A_i = \{j \in \{1, \ldots, n\} : j \preccurlyeq i\}$. It is routine to check that $A_j \subseteq A_i$ if and only if $j \preccurlyeq i$, and $A_j = A_i$ if and only if $j = i$.

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Let χ *denote one of* EPow, DCom, Com, NGen. *Then, given any finite graph* Γ*, there is a group* G *such that* Γ *is isomorphic to an induced subgraph of* $\chi(G)$.

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Question

Given a graph type X*, and a class* $\mathcal{C}$ *of graphs defined by forbidden induced subgraphs, determine the groups* G *for which* $\chi(G) \in \mathcal{C}$*.*

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- If we simply have no green edges, then we have simultaneously embedded the red graph in the enhanced power graph, the deep commuting graph, and the commuting graph.

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- If we simply have no green edges, then we have simultaneously embedded the red graph in the enhanced power graph, the deep commuting graph, and the commuting graph.
- If we ignore the red/blue distinction, we get an embedding of an arbitrary graph in the graph $(\mathrm{Com} - \mathrm{EPow})(G)$ for some group G.

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Second, consider the non-abelian group of order p^3 and exponent p^2, where p is an odd prime:

$$P = \langle a, b \mid a^{p^2} = b^p = 1, [a, b] = a^p \rangle.$$

Any two elements of $\langle a \rangle$ generate a cyclic group; and the group generated by b and x is cyclic if $x = 1$, abelian but not cyclic if $x = a^p$, and non-abelian if $x = a$.

The proof is by induction on the number n of vertices. The result is clearly true if $n = 1$. So let $\{v_1, \ldots, v_n\}$ be the vertex set of Γ, and suppose that we have an embedding of $\{v_1, \ldots, v_{n-1}\}$ into a group G satisfying conditions 1–3 of the theorem.

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If we now embed v_n as $(b, 1)$, we find that the conditions hold for the remaining pairs as well. $\square$

The non-generating graph

Finally we show the same universality property for the non-generating graph. First, we need a preliminary result.

Theorem

Every graph without isolated vertices and edges can be represented as the intersection graph of a family of sets (that is, the vertices are identified with the sets, two vertices adjacent if the corresponding sets have non-empty intersection).

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Proof.

Let E be the edge set of Γ, and for each vertex v, let S_v be the set of edges incident with v. Then

$$S_v \cap S_w = \begin{cases} \{e\} & \text{if } e = \{v, w\}; \\ \emptyset & \text{if } v \text{ and } w \text{ are not joined.} \end{cases}$$

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Step 3 Now replace each set by its complement. The complements of two subsets of Ω have union Ω if and only if the two sets are disjoint. Thus, each original vertex is now represented by an $(n - k)$-set where two such sets have union Ω if and only if the corresponding vertices are adjacent in Γ.

Step 4 Replace each set by a cyclic permutation on that set, fixing the remaining points. Each of these cycles has odd prime length, so each is an even permutation, and so lies in the alternating group A_n. Let g_v be the permutation corresponding to the vertex v of Γ.

- If v and w are nonadjacent, then the supports of g_v and g_w have union strictly smaller than Ω, so $\langle g_v, g_w \rangle \neq A_n$.
- Suppose v and w are adjacent. Then the supports of g_v and g_w have union Ω, so $H = \langle g_v, g_w \rangle$ is transitive on Ω. Using Jordan's theorem, we conclude that H contains the alternating group A_n. Since it is generated by even permutations, $H = A_n$.

Further properties and parameters

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I will not even attempt to summarise all this work. Instead, I will say a small amount about cliques (complete subgraphs) and independent sets (null induced subgraphs). The clique number $\omega(\Gamma)$ is the number of vertices in the largest clique, and the independence number $\alpha(\Gamma)$ is the number of vertices in the largest independent set.

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There are a vast number of graph properties and parameters, many of which have been studied for individual graph types in the hierarchy.

For example, a survey of power graphs in 2013 included nearly 100 references, while a survey of developments since then has another nearly 100 references.

I will not even attempt to summarise all this work. Instead, I will say a small amount about cliques (complete subgraphs) and independent sets (null induced subgraphs). The clique number $\omega(\Gamma)$ is the number of vertices in the largest clique, and the independence number $\alpha(\Gamma)$ is the number of vertices in the largest independent set.

Since power graphs are perfect, the clique number of $\mathrm{Pow}(G)$ is equal to its chromatic number, and the independence number is equal to the clique cover number.

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The only thing that needs comment is that, if a set of elements in a group has the property that any two generate a cyclic group, then the whole set is contained in a cyclic group. The proof is an exercise.

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Theorem

The clique number of $\mathrm{Pow}(C_n)$ *is equal to* $f(n)$.

Proof.

The group C_n has $\phi(n)$ generators; they are dominating vertices in the power graph, so are contained in every maximal clique. It can be shown that the remainder of any maximal clique is contained in a proper subgroup, and the best we can do is to take a maximum-size clique in the largest proper subgroup, the cyclic group of order n/p. Now induction gets us home. □

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In fact, the limit superior of the ratio $f(n)/\phi(n)$ is about $2.6481017597\ldots$. Sean Eberhard has observed that it is equal to

$$\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{1}{p_i - 1},$$

where $p_1, p_2, \ldots$ are the primes in order.

General groups

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The function f is not monotonic, so it is not true that $\omega(\mathrm{Pow}(G)) = f(\omega(\mathrm{EPow}(G)))$. Let $G = \mathrm{PGL}(2, 11)$. The maximal elements of $\omega(G)$ are 10, 11 and 12; so $\omega(\mathrm{EPow}(G)) = 12$. We have $f(10) = f(12) = 9$ and $f(11) = 11$, so $\omega(\mathrm{Pow}(G)) = 11$.

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I will also say something about automorphisms of these graphs.